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Koszul duality

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *This paper tries to describe a natural framework for the canonical equivalence between derived categories of graded modules over symmetric and exterior algebra, which has been established by J. N. Bernstein, I. M. Gelfand and S. I. Gelfand.*

INTRODUCTION

In their beautiful note [BGG1] J. N. Bernstein, I. M. Gelfand and S. I. Gelfand established a canonical equivalence between derived categories of graded modules over symmetric and exterior algebra (which may be called «derived category bonson-fermion equivalence»). Our paper is a mere attempt to describe a natural living area of this subject.

Namely, we consider differential graded algebras that satisfy some natural conditions (we call them mixed algebras). For any mixed algebra A one defines the new mixed algebra $\check{A}$, called Koszul dual to A (the cohomology of $\check{A}$ equals to Ext's of A with simple coefficients). The derived categories $D(A), D(\check{A})$ (of differential graded modules) are canonically equivalent («Koszul duality»). The Koszul dual to a symmetric

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algebra is an exterior one, whence BGG equivalence. Here also the Koszul quadratic algebras, introduced by Priddy [P], naturally arise.

What are the most interesting graded algebras? The vast important supply comes from geometry of mixed sheaves. Namely, the category of mixed perverse sheaves, lisse along certain stratification, is naturally equivalent to some category of graded modules (this explains our attachment to the term «mixed» for such sort of categories). Unfortunately, we do not know any geometric interpretation of Koszul duality at a moment.

How to construct these «geometric» algebras explicitly? The very intriguing subject is the case of Schubert stratification. Here one gets the Koszul quadratic algebra which seems very much to be self-dual. The corresponding category is the mixed version of Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [BGG2] that lies in the origin of Kazhdan-Lusztig algorithm. In the next paper we plan to apply the technique developed below to mixed representations.

Note also that there is a variety of extremely beatiful algebras of Yang-Baxter breed (see e.g. [D], [Ma]); it would be very interesting to relate them with «geometric» algebras.

The paper goes as follows. The §1 contains some preliminaries on homological algebra in the spirit of [BBD], §1 – 3; it has some overlaps with [BK]. The §2 treats the Koszul duality in the context of filtered t-categories. The same subject is considered in §3 from differential algebras view-point; this section was influenced by C. Moore's report [Mo]. In §4 we consider the important particular case of quadratic algebras, and §5 contains some easy examples.

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NOTATIONS AND CONVENTIONS

0.1 For a category C $x \in C$ means $x \in ObC$; C^{opp} denotes the opposed category. A full subcategory $D \subset C$ is called *strictly full* if $x \in D, y \cong x$ implies $y \in D$.

0.2 Let $D \subset C$ be a subcategory of an additive category. $D^{\perp}$ (resp. ${}^{\perp}D$) is the full subcategory of C whose objects are such $x \in C$ that for every $y \in D$

$\mathrm{Hom}(y, x) = 0$ (resp., $\mathrm{Hom}(x, y) = 0$).

0.3 For an additive category $AC^{\cdot}(A)$ denotes the category of complexes over A. For $a \in \mathbb{Z}C^{\leq a}(A)$ (resp., $C^{\geq a}(A)$) is in full subcategory of such $x \in C(A)$ that $x^i = 0$ for $i > a$ (resp., $i < a$). $C^{a,b}(A) = C^{\geq a}(A) \cap C^{\leq b}(A)$; $C^a(A) := C^{a,a}(A)$. We also use sometimes homological notation: for $x \in C^{\cdot}(A)$ we put $x_i := x^{-i}$.

$Gr(A)$ is the category of graded objects over A (a full subcategory of $C^{\cdot}(A)$ of complexes with zero differential); $Gr^*(A) = Gr(A) \cap C^*(A)$ where $* = +, -, b$, are standard boundedness notations.

For a ring k $M(k) :=$ a category of left k-modules; $C^{\cdot}(k) = C^{\cdot}(M(k))$, etc.

0.4 Let $(D, D^{\leq 0}, D^{\geq 0})$ be a t-category, [BBD], 1.3.1. We put $D^{a,b} := D^{\geq a} \cap D^{\leq b}$, $D^a := D^{a,a} (a, b \in \mathbb{Z})$. In particular, D^0 is core of t-structure.

Recall that a t-structure is t-non-degenerate if the following equivalent conditions hold

(i) $\bigcap_a D^{\leq a} = \bigcap_a D^{\geq a} = 0$;

(ii) An object $x \in D$ equals zero iff it is acyclic i.e. $H^a(x) = 0$ for any $a \in \mathbb{Z}$.

A t-structure is called t-bounded if the following equivalent conditions hold:

(i) $\bigcup_a D^{\leq a} = \bigcup_a D^{\geq a} = D$;

(ii) The t-structure is t-non-degenarate and any $x \in D$ has only finitely many non-zero $H^a(x)$.

(iii) D is generated (as triangulated category) by its core.

If $(D, D^{\leq 0}, D^{\geq 0})$ is a t-category, $n \in \mathbb{Z}$, then $D[n]$ denotes t-category $(D, D^{\leq 0}[n], D^{\geq 0}[n])$.

0.5 Consider a sequence of categories and functors

$$(*) \qquad C_1 \xrightarrow{i} C \xrightarrow{p} C_2$$

We will say that (*) is *exact sequence* if either

– C's are abelian categories: then i is equivalence of C_1 with Serre subcategory of C and p is equivalence of the corresponding quotient category with C_2.

– C's are triangulated: then i is equivalence of C_1 with saturated subcategory of C and p is equivalence of the quotient category with C_2.

If C's are t-categories we will say that (*) is *t-exact* if it is exact and both i and p are t-exact functors.

0.5.1 LEMMA. *If (*) is t-exact sequence, then one has*

$$(C_1^{\leq 0}, C_1^{\geq 0}) = (i^{-1}C^{\leq 0}, i^{-1}C^{\geq 0}), \quad (C_2^{\leq 0}, C_2^{\geq 0}) = (pC^{\leq 0}, pC^{\geq 0})$$
$$(C^{\leq 0}, C^{\geq 0}) = ({}^{\perp}i(C_1^{>0}) \cap p^{-1}(C_2)^{\leq 0}), i(C_1^{<0})^{\perp} \cap p^{-1}(C_2)^{\geq 0}).$$

∎

This means that if (*) is exact sequence of triangulared categories, then

– given t-structures on C_1, C_2 there exists at most one t-structure on C such that (*) is t-exact.

– given t-structure on C one has at most one t-structures on C_1, C_2 such that (*) is t-exact. In fact, to have them it sufficies to verify that either $(i^{-1}C^{\leq 0}, i^{-1}C^{\geq 0})$ is a t-structure on C_1 or $(pC^{\leq 0}, pC^{\geq 0})$ is a t-structure on C_2. If so we will say that these t-structures are *induced* by the one on C.

If (*) is t-exact sequence, then the sequence of cores $M_1 \to M \to M_2$ is also exact. Conversely, given a bounded t-category$C$ with core M, and $M_1 \to M \to M_2$ is exact sequence, then it comes from unique t-exact sequence $C_1 \to C \to C_2$ namely; C_1 is triangled subcategory generated by M_1, $C_2 = C/C_1$.

If (*) is t-exact sequence, then C is bounded iff both C_1, C_2 are; same for non-degenerate property.

1. HOMOLOGICAL PRELIMINARIES

1.1 Semisimple triangulated categories

1.1.1 Recall that an abelian category A is called *semisimple* if every short exact sequence in A splits. If every object in A has finite lenght then A is semisimple iff every its object is a direct sum of simple ones.

We denote by $\operatorname{Irr} A$ the set of isomorphism classes of simple objects of A.

1.1.2 Let D be a triangulated category. Call a distinguished triangle in D *split* if it is isomorphic to a triangle of the form

(1.1.2.1) $$a \oplus b \to b \oplus c \to c \oplus a[1] \to a[1] \oplus b[1]$$

where all arrows are $x \oplus y \overset{0\oplus \mathrm{id}_y}{\longrightarrow} z \oplus y \overset{\mathrm{can}}{\longrightarrow} y \oplus z$.

1.1.3 LEMMA-DEFINITION. *Let D be a trianguled category. The following conditions are equivalent:*

(a) D is abelian.

(b) D is abelian semisimple.

(c) Every arrow in D is isomorphic to an arrow of the form

$$\mathrm{id}_a \oplus 0 : a \oplus b \to a \oplus c.$$

(d) Every distinguished triangle in D is split.

When these conditions are satisfied, we say that D is semisimple. ∎

1.1.4 Let A be an abelian semisimple category and $[1] : A \to A$ an automorphism. Let us call a distinguished triangle in A a triangle isomorphic to (1.1.2.1). This defines on A a structure of a semisimple triangulated category and every semisimple triangulated category comes this way.

1.1.5 EXAMPLE. Let A be a semisimple abelian category. Then $D^*(A)(* = \emptyset, +, -, b)$ is equivalent to $Gr^*(A)$; it is semisimple triangulated category.

In fact, every $x \in D$ is canonically isomorphic to $\oplus H^i(x)$.

1.1.6 LEMMA. *Let D be a semisimple triangulated category equipped with a bounded t-structure with the core A. Then D is canonically equivalent to $D^b(A)$.*

1.2 Steps and ladders

1.2.1 DEFINITION. *A step* is a triple of triangulated categories connected with exact functors

$$U \underset{\alpha^{\cdot}}{\overset{\alpha_{\cdot}}{\rightleftharpoons}} D \underset{\beta_{\cdot}}{\overset{\beta^{\cdot}}{\rightleftharpoons}} V \tag{1.2.1.1}$$

such that

– $\alpha_{\cdot}$ (resp., $\beta^{\cdot}$) is left adjoint to $\alpha^{\cdot}$ (resp., $\beta_{\cdot}$);

– adjunction maps $\mathrm{Id}_U \to \alpha^{\cdot}\alpha_{\cdot}$ and $\beta^{\cdot}\beta_{\cdot} \to \mathrm{Id}_V$ are isomorphisms (i.e. $\alpha_{\cdot}$ and $\beta_{\cdot}$ are faithfully full embeddings).

– $\alpha_{\cdot}(U) = {}^{\perp}(\beta_{\cdot}(V))$; $\beta_{\cdot}(V) = (\alpha_{\cdot}(U))^{\perp}$ (consequently, $\alpha_{\cdot}(U)$ and $\beta_{\cdot}(V)$ are strictly full saturated subcategories of D);

– $\alpha^{\cdot}$ (resp., $\beta^{\cdot}$) induces equivalence $D/\beta_{\cdot}(V) \xrightarrow{\sim} U$ (resp., $D/\alpha_{\cdot}(U) \xrightarrow{\sim} V$).

Clearly $U \xrightarrow{\alpha_{\cdot}} D \xrightarrow{\beta^{\cdot}} V$ and $V \xrightarrow{\beta_{\cdot}} D \xrightarrow{\alpha^{\cdot}} U$ are exact sequences (0.5), and the step is uniquely determined by either of them.

Usually we will identify U, V with their images in D.

Sometimes we'll draw a step (1.2.1.1.) in a reflected shape

$$V \underset{\beta_{\cdot}}{\overset{\beta^{\cdot}}{\rightleftarrows}} D \underset{\alpha^{\cdot}}{\overset{\alpha_{\cdot}}{\rightleftarrows}} U \tag{1.2.1.1^r}$$

Note that because of the direction of arrows no ambiguity arises.

In the opposed category we get *the opposed step*:

$$V^{\mathrm{opp}} \underset{\beta^{\cdot\,\mathrm{opp}}}{\overset{\beta_{\cdot}^{\mathrm{opp}}}{\rightleftharpoons}} D^{\mathrm{opp}} \underset{\alpha_{\cdot}^{\mathrm{opp}}}{\overset{\alpha^{\cdot\,\mathrm{opp}}}{\rightleftharpoons}} U^{\mathrm{opp}} \tag{1.2.1.2}$$

An n-step ladder is a diagram of the form

$$\begin{array}{ccccc} & \overset{\varphi_1}{\longrightarrow} & & \overset{\psi_1}{\longrightarrow} & \\ & \overset{\varphi_2}{\longleftarrow} & & \overset{\psi_2}{\longleftarrow} & \\ & \overset{\varphi_3}{\longrightarrow} & & \overset{\psi_3}{\longrightarrow} & \\ U & \vdots & D & \vdots & V \\ & \overset{\varphi_{n+1}}{\longleftarrow} & & \overset{\psi_{n+1}}{\longleftarrow} & \end{array} \quad \text{for } n \text{ odd},$$

or

$$\begin{array}{ccccc} & \overset{\varphi_1}{\longrightarrow} & & \overset{\psi_1}{\longrightarrow} & \\ & \overset{\varphi_2}{\longleftarrow} & & \overset{\psi_2}{\longleftarrow} & \\ & \overset{\varphi_3}{\longrightarrow} & & \overset{\psi_3}{\longrightarrow} & \\ U & \vdots & D & \vdots & V \\ & \overset{\varphi_{n+1}}{\longleftarrow} & & \overset{\psi_{n+1}}{\longleftarrow} & \end{array} \quad \text{for } n \text{ even},$$

such that for every i $(\varphi_i, \varphi_{i+1}; \psi_i, \psi_{i+1})$ forms a step (in the form $(1.2.1.1)^r$ for i even).

1.2.2 If we have a step 1.2.1 then for every $x \in D$ adjunction maps for an exact triangle

$$\alpha_. \alpha^{\cdot} x \to x \to \beta_. \beta^{\cdot} x$$

Conversely,

1.2.3 LEMMA. (cf. [BBD], 1.4.4., [V], §2, n. 3). *Let $U, V \subset D$ be strictly full triangulated subcategories of a triangulated category, such that $U \subset^{\perp} V$ and for every $x \in D$ there exists a distinguished triangle*

(1.2.3.1) $$u \to x \to v \to u[1]$$

with $u \in U, v \in V$. Then

(a) (1.2.3.1) is unique up to a canonical isomorphism

(b) a diagram of inclusions

$$U \overset{\alpha_.}{\to} D \underset{\beta_.}{\leftarrow} V$$

may be uniquely completed to a step, i.e. $\alpha_.$ (resp., $\beta_.$) admit right (resp., left) adjoint $\alpha^{\cdot}(\beta^{\cdot})$ satisfying 1.2.1 ∎

We leave the proof the reader; note only that for $x \in D$ $\alpha^{\cdot}(x)$ and $\beta^{\cdot}(x)$ are just u and v in (1.2.3.1).

1.2.4 COROLLARY. *Let $U \to D$ be a strictly full triangulated subcategory. Suppose that for every $x \in D$ there exists a distinguished triangle $u \to x \to v \to u[1]$ (resp., $v \to x \to u \to v[1]$) with $u \in U, v \in U^{\perp}$ (resp., $v \in^{\perp} U$). Then $U \to D \leftarrow U^{\perp}$ (resp. $^{\perp}U \to D \leftarrow U$) may be completed to a step*

$$U \rightleftarrows D \rightleftarrows U^{\perp} \quad (\text{resp.}\, ^{\perp}U \rightleftarrows D \rightleftarrows U).$$ ∎

1.3 Filtered triangulated categories

1.3.1 DEFINITION. *A filtered triangulated category*, or f-*category* for short, is a triangulated category D together with a set of strictly full triangulated subcategories $D_{\leq a}$, $D_{\geq a}$, $a \in \mathbb{Z}$, such that

(F1) for all $a \in \mathbb{Z}$ $D_{\leq a} \subset^{\perp} D_{\leq a+1}$ and $D_{\geq a} \supset D_{\geq a+1}$;

(F2) for all a $a \in \mathbb{Z}$ $D_{\leq a} \subset D_{>a}$ (here $D_{>a} := D_{\geq a+1}$);

(F3) for all $a \in \mathbb{Z}, x \in D$, x may be included in an exact triangle

$$(1.3.1.1) \qquad y \to x \to z \to y[1], \qquad y \in D_{\leq a}, z \in D_{>a}$$

We also call this data *an f-structure* on D. We will say that D (or f-structure) is *f-non-degenerate* if $\bigcap_a D_{\leq a} = \bigcap_a D_{\geq a} = 0$, and *$f$-bounded* if $D = \bigcup_a D_{\geq a} = \bigcup_a D_{\leq a}$. Any f-bounded category is f-non-degenerate.

Let C, D be f-categories, and $F : C \to D$ be exact functor; we will say that F is *f-exact* if $F(C_{\leq a}) \subset D_{\leq a}$, $F(C_{\geq a}) \subset D_{\geq a}$ for any $a \in \mathbb{Z}$.

If D is f-category and $n \in Z$ we denote by $D(n)$ – the f-shift by n– the f-structure on D given the formula $(D(n)_{\leq a}, D(n)_{\geq a}) = (D_{\leq a+n}, D_{\geq a+n})$, and by D^{opp} the f-category $(D^{\text{opp}}, D^{\text{opp}}_{\leq a}, D^{\text{opp}}_{\geq a}), D^{\text{opp}}_{\leq a} := (D_{\geq -a})^{\text{opp}}$.

Let $(D, D_{\leq a}, D_{\geq a})$ be an f-category. Then for all a pair $(D_{\leq a}, D_{\geq a})$ satisfies the assumptions of 1.2.3. Functors $D \to D_{\leq a}, D \to D_{\geq a}$ adjoint to inclusions we shall denote $w_{\leq a}, w_{\geq a}$ and call functors of *truncation by filtration*; by 1.2.3 they are exact.

For $a, b \in \mathbb{Z}$ put $D_{a,b} = D_{\geq a} \cap D_{\leq b}$; $D_a := D_{a,a}$; for every a, b $D_{a,b}$ with the induced filtration $(D_{a,b} \cap D_{\leq i}; \mathcal{D}_{a,b} \cap \mathcal{D}_{\geq i})$ forms an f-category.

1.3.2 LEMMA. *For all* $a \leq b$

(a) one has canonical isomorphism of functors

$$w_{\geq a} w_{\leq b} \tilde{\to} w_{\leq b} w_{\geq a} : D \to D_{a,b};$$

(b) canonical projections induce equivalences

$$D_{\leq b}/D_{\leq a-1} \tilde{\leftarrow} D_{a,b} \tilde{\to} D_{\geq a}/D_{\geq b+1}$$

■

We put $gr_{a,b} = w_{\geq a} w_{\leq b} = w_{\leq b} w_{\geq a}$; $gr_a = gr_{a,a}$.

1.3.3 Let be an exact sequence (*) (0.5) of triangulated categories. Assume that C, C_i are equipped with f-structures. We will say that (*) if *f-exact* if i, p are f-exact functors. In this case one has (cf. 0.5)

$$(C_{1\leq a}, C_{1\geq a}) = (i^{-1}C_{\leq a}, i^{-1}C_{\geq a}),$$

$$(C_{2\leq a}, C_{2\geq a}) = (pC_{\leq a}, pC_{\geq a})$$

$$(C_{\leq a}, C_{\geq a}) = (p^{-1}C_{2\leq a} \cap {}^{\perp}iC_{1>a}, p^{-1}C_{2\geq a} \cap (iC_{1<a})^{\perp})$$

An f-exact sequence (*) defines exact sequences

$$(*)_a \qquad C_{1a} \to C_a \to C_{2a}$$

Conversely, assume that C is bounded f-category. Then any set $(*)_a$ of exact sequences comes from unique f-exact sequence (*) (just put C_1 to be the triangulated subcategory in C generated by $\{iC_{1a}\}$).

REMARK. In [B1] the term « f-category » was used for the f-data with some additional structures.

1.3.4 Finiteness axiom

Consider the following conditions on a f-category D :

(FR) for every i the functor $w_{\leq i} : D \to D_{\leq i}$ admits a right adjoint $I_i : D_{\leq i} \to D$;

(FL) for every i the functor $w_{\geq i} : D \to D_{\geq i}$ admits a left adjoint $P_i : D_{\geq i} \to D$;

(F) =(FR) & (FL).

Usually the weaker axioms hold:

(FR)′ for every $a, b \in \mathbb{Z}$ the f-category $D_{a,b}$ satisfies (FR). Similarly for (FL)′, (F)′.

1.3.4.1 LEMMA. *FR implies $(FR)'$: the functor I_i for $D_{a,b}$ equals to $w_{\leq b} I_i|_{D_{a,i}}$. Same holds for (FL).* ∎

COROLLARY. *Assume that $(FR)'$ holds. Then the functors $I_{(a,i,b)} : D_{a,i} \to D_{a,b}$ adjoint to $w_{\leq i}$ are compatible: for $a \leq a' \leq i \leq b' \leq b$ one has*

$$I_{a,i,b}(D_{a',i}) \subset D_{a',b};$$

$$I_{a,i,b}|_{D_{a',i}} = I_{a',i,b}; \qquad w_{\leq b'} I_{a,i,b} = I_{a,i,b'};$$

same for P. ∎

1.3.4.2 LEMMA. *If D satisfies (FR) (resp., (FL) then a canonical step*

$$D_{\leq i} \underset{w_{\leq i}}{\rightleftarrows} D \overset{w_{\leq i}}{\rightleftarrows} D_{\geq i+1}$$

may be embedded in a two-step ladder

$$\begin{array}{ccccc} & \longrightarrow & & \overset{w_{\geq i}}{\longrightarrow} & \\ D_{\leq i} & \overset{w_{\leq i}}{\longleftarrow} & D & \longleftarrow & D_{\leq i+1} \\ & \overset{I_i}{\longrightarrow} & & \longrightarrow & \end{array}$$

(resp.,

$$D_{\le i} \; \underset{\overset{\longleftarrow}{w_{\le i}}}{\overset{\longleftarrow}{\longrightarrow}} \; D \; \underset{\longleftarrow}{\overset{\overset{P_i}{\longleftarrow}}{\overset{w_{\ge i}}{\longrightarrow}}} \; D_{i+1})$$ ■

1.3.4.3 LEMMA. *Assume that* D *is an* f*-category such that all* D_a *are semisimple and each object of* D_a *has finite length. Then* (FR)$'$ *property is equivalent to the following conditions:*

$(\mathrm{i})_R$ *for every* a,b *and* $x \in \operatorname{Irr} D_a, y \in \operatorname{Irr} D_b$ $\operatorname{Hom}(y,x)$ *is End* y*-module of finite length;*

$(\mathrm{ii})_R$ *for every* a,b *and* $x \in \operatorname{Irr} D_a$ *there exist only finitely many* $y \in \operatorname{Irr} D_b$ *such that* $\operatorname{Hom}(y,x) \neq 0$.

Similarly, (FL)$'$ *is equivalent to* $(\mathrm{i})_L, (\mathrm{ii})_L$, *obtaines from* $(\mathrm{i})_R, (\mathrm{ii})_R$ *replacing* $\operatorname{Hom}(y,x)$ *by* $\operatorname{Hom}(x,y)$.

Proof. We restrict ourselves to the implication $(\mathrm{i})_R \& (\mathrm{ii})_R \Rightarrow$ (FR). First suppose that $i = b-1$. For $x \in D_{a,i}$ put $u = \bigoplus_{y \in \operatorname{Irr} D_b} (\operatorname{Hom}_D(y,x)_{End y} \otimes y) \in D_b$ (cf. 1.1.1.2). Let $f: u \to x$ be a canonical morphism, and

(*) $$u \xrightarrow{f} x \to v \to u[1]$$

some exact triangle. One verifies that $v \in D_b$, thus by 1.2.3 (*) is unique; so for $x \in D_{a,i}$ we put $I_i(x) = v$. Iterating this construction, we obtain I_i for every i.

Dually, P is constructed for $i = a$ using canonical map $g: x \to \bigoplus_{y \in \operatorname{Irr} D_a} \operatorname{Hom}_{\mathrm{End}\, y} (\operatorname{Hom}_D(x,y), y)$. ■

1.3.5 Dual filtration

Let D be an f-category satisfying (FR).

For $a \in \mathbb{Z}$ put

$$D^V_{\le a} = D_{\ge -a}; \qquad D^V_{\ge a} = I_{-a}(D_{\le -a})$$

Similarly, if (FL) holds then put

$${}^V D_{\le a} = P_{\ge -a}(D_{\ge -a}), \qquad {}^V D_{\ge -a} = D_{\le -a}.$$

From 1.3.4.2 immediately follows that in such a way we get new f – structures on D, which we call right and left dual to the initial one.

Clearly, we have $D = {}^V(D^V) = ({}^V D)^V$.

1.3.6 Canonical resolutions

1.3.6.1 Let us call a complex $\ldots \to x^i \xrightarrow{d} x^{i+1} \xrightarrow{d} \ldots$ (a sequence of morphisms with $d^2 = 0$) in a triangulated category *exact* if it may be embeded into a caterpillar diagram

$$\begin{array}{ccccccccccccc} \ldots \to & & x^i & & \to & & x^{i+1} & & \to & & x^{i+2} & & \to \ldots \\ & C & \nearrow & E & \searrow & C & \nearrow & E & \searrow & C & \nearrow & E & \searrow \quad C \\ & & \dot{\leftarrow} & & y^i & & \dot{\leftarrow} & & y^{i+1} & & \dot{\leftarrow} & & y^{i+2} \end{array}$$

where $\dot{\leftarrow}$ denotes a morphism of degree 1, in which all triangules C are commutative and E exact.

1.3.6.2 Let D be an f-category satisfying $(F)'$. Fix $a, b \in \mathbb{Z}$, $a \le b$. For $a \le i \le b$ put $Pg_i = P_i \dot{g} r_i$; $Ig_2 = I_i \dot{g} r_i$ (cf. 1.3.3). We have for $x, y \in D_{a,b}$

$$\mathrm{Hom}_{D_{a,b}}(Pg_i(x), y) = \mathrm{Hom}_{D_i}(gr_i x, gr_i y) = \mathrm{Hom}_{D_{a,b}}(x, Ig_i(y)).$$

Put $P_0(x) = \bigoplus_{a\le i\le b} Pg_i(x)$. Let $P_0(x) \to x$ be a canonical map induced by ϵ_i : $Pg_i(x) \to x$ corresponding to id : $gr_i x \to gr_i x$. More generally, for $0 \le r \le b-a$ put

$$P_r(x) = \bigoplus_{a\le i_0<\ldots<i_r\le b} Pg_{i_0} Pg_{i_1} \ldots Pg_{i_r}(x)$$

For $0 \le j \le r$ we have maps

$$Pg_{i_0} Pg_{i_{01}} \ldots Pg_{i_r} \to Pg_{i_0} \ldots \widehat{Pg_{i_0}} \ldots Pg_{i_r}$$

equal to $Pg_{i_0} \ldots Pg_{i_{j-1}} \dot{\epsilon}_{i_j} \dot{P}g_{i_{j+1}} \ldots Pg_{i_r}$. They induce $\partial_j : P_r(x) \to P_{r-1}(x)$; put $d = d^r = \sum_{j=0} (-1)^j \partial_j : P_r \to P_{r-1}$. So we get a complex in $D_{a,b}$

$$\overline{P}_{\cdot} = P_{\cdot}(x) \to x : 0 \to P_{b-a}(x) \to \ldots \to P_0(x) \to x \to 0$$

Dually, put

$$I^{\cdot}(x) = \bigoplus_{a\le i_0<\ldots<i_r\le b} Ig_{i_r} \circ Ig_{i_{r-1}} \circ \ldots \circ Ig_{i_0}(x)$$

and define

$$\overline{I}^{\cdot} = x \to I^{\cdot}(x) : 0 \to x \to I^0(x) \to \ldots \to I^{b-a}(x) \to 0$$

1.3.6.3 LEMMA. *These complexes are exact.*

Proof. Induction on $b-a$, using 1.3.4.1 For example, we have $w_{\geq b}\bar{P}^{\cdot}(x) = (\ldots 0 \to w_{\geq b}x = w_{\geq b}x \to 0)$; and $w_{\leq b-1}(\bar{P}_{cdot}(x))$ may be included in an exact triangle

$$\bar{P}^{\cdot}(w_{\leq b-1}x) \to w_{\leq b-1}\bar{P}^{\cdot}(x) \to \bar{P}^{\cdot}(w_{\leq b-1}Pg_b(x)).$$ ∎

1.4 Filtered t-categories

1.4.1 LEMMA-DEFINITION. *Let $(D; D_{\leq a}, D_{\geq a})$ be a filtered triangulated category, $(D^{\leq 0}, D^{\geq 0})$ – a t-structure on D. The following conditions are equivalent*

(a) For each a the categories $D_{\leq a}, D_{\geq a}$ carry a (unique) t-structures such that both canonical sequences $D_{\leq a} \to D \to D_{>a}$, $D_{>a} \to D \to D_{\leq a}$ are t-exact;

(b) all $w_{\geq a}, w_{\leq a}$ are t-exact.

When these conditions are satisfied we say that the t-structure is compatible with the f-structure, and D a filtered t-category, *or* ft-category *for short.* ∎

1.4.2 LEMMA. *Let D be an ft-category, M be core of D.*

(i) *For any $x \in D$ there exists unique isomorphism between canonical diagrams*

$$\begin{array}{ccccc}
w_{\leq i}\tau_{>a}x & \to & \tau_{>a}x & \to & w_{>i}\tau_{>a}x \\
\uparrow & & \uparrow & & \uparrow \\
w_{\leq i}x & \to & x & \to & w_{>i}x \\
\uparrow & & \uparrow & & \uparrow \\
w_{\leq i}\tau_{\leq a}x & \to & \tau_{\leq a}x & \to & w_{>i}\tau_{\leq a}x
\end{array}$$

and

$$\begin{array}{ccccc}
\tau_{>a}w_{\leq i}x & \to & \tau_{>a}x & \to & \tau_{>a}w_{>i}x \\
\uparrow & & \uparrow & & \uparrow \\
w_{\leq i}x & \to & x & \to & w_{>i}x \\
\uparrow & & \uparrow & & \uparrow \\
\tau_{\leq a}w_{\leq i}x & \to & \tau_{\leq a}x & \to & \tau_{\leq a}w_{>i}x
\end{array}$$

that induces identity on the central cross. Hence on vertices we get canonical isomorphisms of functors $\tau_{\leq a}w_{\geq i} = w_{\geq i}\tau_{\leq a}$, etc.

(ii) *The functors gr_a are t-exact.*

(iii) *One has $w_{\leq a}M, w_{\geq a}M \subset M$.*

(iv) *For any $x \in M$ one has canonical short exact sequence $0 \to w_{\leq a}x \to x \to w_{>a}x \to 0$; hence the objects $x \in M$ are equipped with the canonical filtration $w_{\leq a}x$ and any morphism in M is strictly compatible with this filtration.* ∎

REMARKS. (i) If D is f-bounded, then 1.4.2 (ii) is equivalent to conditions 1.4.1; if D is t-bounded then 1.4.2 (iii) is equivalent to 1.4.1.

(ii) If D is an ft-category then so are all $D_{a,b}$ with induced structure.

(iii) We will call an ft-category *ft-bounded* if it is both f- and t-bounded; a functor between ft-categories is *ft-exact* if it is both f- and t-exact.

(iv) Consider an exact sequence (*) (0.5) of triangulated categories equipped with f-and t-structure s; assume that is both f-and t-exact. Then if C is ft-category then both C_1, C_2 are.

1.4.3 DEFINITION. *A filtered* (or *f-*) *abelian category* is an abelian category M such that each object X M carries a canonical increase filtration $w_\cdot, w_i(X) = X_{\leq i}$ such that any morphism is strictly compatible with $w_\cdot$.

We will also call this data *an f-structure* on abelian category M. The equivalent way to define f-structure is to give two chains of full subcategories $\ldots \subset M_{\leq i} \subset M_{\leq i+1} \subset \ldots$, $\ldots \supset M_{\geq i} \supset M_{\geq i+1} \supset \ldots$, of M such that $M_{\leq i} = {}^{\perp} M_{\geq i+1}$, $M_{\geq i} = M^{\perp}_{\leq i-1}$ such that for any $X \in M, i \in \mathbf{Z}$ there exists a short exact sequence $0 \to X_{\leq i} \to X \to X_{\geq i+1} \to 0$, $X_{\leq i} \in M_{\leq i}, X_{\geq i+1} \in M_{\geq i+1}$ (one has $X_{\leq i} = w_i(X)$, and $M_{\leq i}$ consists of X 's with $w_i(X) = X$). Say that M is *f-bounded* if $M = \cup M_{\leq i} = \cup M_{\geq i}$ and *f-nondegenerate* if $\cap M_{\leq i} = \cap M_{\geq i} = 0$. If $F : M \to N$ is a functor between f-abelian categories we say that F is *f-exact* if it is exact and $F(M_{\leq i}) \subset N_{\leq i}$, $F(M_{\geq i}) \subset N_{\geq i}$. One has

(i) If D is ft-category, then its core M is f-abelian category with $M_{\leq i} = D_{\leq i} \cap M$.

(ii) If M is f-abelian category, then $D^{\cdot}(M)$ is ft-category (the t-structure is standard, $D^{\cdot}(M)_{\leq i} := D^{\cdot}(M_{\leq i})$)

(iii) For a t-bounded t-category D with core M, (i) defines 1-1 correspondence between ft-structure s on D and f-structures on M (since $D_{\leq i}$ is generated by $M_{\leq i}$) which preserves the properties of being f-bounded and f-nondegenerate.

(iv) Let $M_i \xrightarrow{i} M \xrightarrow{p} M_2$ be a short exact sequence of abelian categories. If M, M_i are equipped with f-structures, then say that it is *f-exact* if both i, p are f-exact functors. Any f-structure on M defines (uniquely) the f-structures on M_1, M_2 such that the sequence is f-exact; call them the induced structures. Similarly, let $D_1 \xrightarrow{i} D \xrightarrow{p} D_2$ be a short exact sequence of categories. If D, D_i are ft-categories, then we say that it is ft-exact if it is both f-and t-exact. If D is a t-bounded ft-category, then it defines (uniquely) the ft-structures on D_i such that the sequence is ft-exact.

1.4.4 EXAMPLE. Let A be an abelian category, $D = DF(A)$ its bounded filtered derived category ([I] ch. V, §1, [BBD], 1.1.4 Example 1); we consider finite increasing filtrations $\ldots \subset F_a \subset F_{a+1} \subset \ldots$. Now D has two natural f-structures $({}^{\wedge}D_{\leq a}, {}^{\wedge}D_{\geq a})$ and $({}^{S}D_{\leq a}, {}^{S}D_{\geq a})$, given by formulas ${}^{\wedge}D_{\leq a} := \{X : \mathrm{gr}^F_i(X) = 0 \text{ for } i > a\}$,

${}^{\Lambda}D_{\geq a} := \{X : \mathrm{gr}_i^F(X) = 0 \text{ for } i < a\}$, ${}^{S}D_{\leq a} := \{X : F_i(X) = 0 \text{ for } i < -a\}$, ${}^{S}D_{\geq a} := \{X : F_i(X) = 0 \text{ for } i > -a\}$. Note that the Λ f-structure is f-bounded, while the S one is not (it is bounded from the right and non-degenerate). One has $D^S_{\leq a} = {}^{\Lambda}D_{\geq -a}$, $D^S_{\geq a} \subset {}^{\Lambda}D_{\leq -a}$, in particular the S f-structure is right dual to the Λ f-structure (1.3.5).

Also D has two natural t-structures (in fact, D is equivalent to bounded derived category of the corresponding cores). The first one $({}^{\Lambda}D^{\leq 0}, {}^{\Lambda}D^{\geq 0})$ is ${}^{\Lambda}D^{\leq 0} := \{X : H^i(\mathrm{Gr}_j X) = 0 \text{ for } i > -j\}$, ${}^{\Lambda}D^{\geq 0} := \{X : H^i(\mathrm{Gr}_j X) = 0 \text{ for } i < -j\}$; its core coincides with the category $C^b(A)$ embedded in D by formula $C^{\cdot} \mapsto (C^{\cdot}, f_a)$, where F_a is stupid filtration: $F_a(C^{\cdot})^i = 0$ for $i < -a$, $F_a(C^{\cdot})^i = C^i$ for $i \geq -a$ (cf. [BBD], 3.1.7). The second t-structure $({}^{S}D^{\leq 0}, {}^{S}D^{\geq 0})$ was defined in [BBD], (1.3. 2.3.); its core is category of generalized filtered objects in A. It is easy to see that Λ f-and t-structures are compatible; same holds for the S ones. Both ft-structures have natural twists by 1: in Λ case it is $\Phi_\Lambda : (X, F_a) \mapsto (X[1], G_a), G_a = F_{a-1}[1]$, in S case it is $\Phi_S : (X, F_a) \mapsto (X, G_a), G_a = F_{a-1}, \Phi_\Lambda = \Phi_S[1]$.

This is a starting example of Koszul dual ft-structures (see 3.6.5).

1.5 Glueing

Let

$$V \;\begin{matrix} \xrightarrow{j_!} \\ \xleftarrow{j^! = j^*} \\ \xrightarrow{j_*} \end{matrix}\; D \;\begin{matrix} \xrightarrow{i^*} \\ \xleftarrow{i_* = i_!} \\ \xrightarrow{i^!} \end{matrix}\; U$$

be a two-step ladder. Consider the corresponding exact sequence

$$(*) \qquad U \xrightarrow{i_*} D \xrightarrow{j^*} V$$

of exact categories.

1.5.1 LEMMA. (i) *For any t-structures on U, V there exists (a unique) t-structure on D such that (*) is t-exact (see 0.5.1).*

(ii) *For any f-structures on U, V there exists (a unique) f – structure on D such that (*) is f-exact (see 1.3.3).* ∎

Here (i) is [BBD], 1.4.10, 1.4.12; (ii) may be proved in a similar way.

1.5.2 Let D be an ft-category satisfying (F). Consider the following diagram

$$(1.5.2.1) \qquad D_{\leq a} \;\begin{matrix} \xleftarrow{j_a^?} \\ \xrightarrow{j_{a!} = j_{a?}} \\ \xleftarrow{j_a^* = j_a^!} \\ \xrightarrow{j_{a*}} \end{matrix}\; D \;\begin{matrix} \xleftarrow{i_{a?}} \\ \xrightarrow{i_a^* = i_a^?} \\ \xleftarrow{i_{a!} = i_{a*}} \\ \xrightarrow{i_a^!} \end{matrix}\; D_{\geq a}$$

Here $j_{a!}, i_{a!}$ are natural inclusions; $j_a^* = w_{\leq a}$; $i_a^* = w_{\geq a}$; $j_{a*} = I_a, i_{a?} = P_a$, etc.
Put $k_{a\cdot} = i_{a\cdot} j_{a\cdot} : D_a \to D_{\geq a} \to D$; $k_a^{\cdot} = j_a^{\cdot} i_a^{\cdot} : D \to D_a$, $\cdot = \star, !, or ?$.
Our t-structure on D induces t-structures on all D_a, see 1.4.4.
Conversely, let $p : \mathbf{Z} \to \mathbf{Z}$ be an arbitrary function («perversity»). Put

$$D^{p\leq 0} = \bigcap_a k_a^{*-1}(D_a^{\geq p(a)}), \quad D^{p\geq 0} = \bigcap_a k_a^{!-1}(D_a^{\geq p(a)});$$

dually, put

$$P_D^{p\leq 0} = \bigcap_a k_a^{?-1}(D_a^{\leq p(a)}), \quad P_D^{\geq 0} = \bigcap_a k_a^{*-1}(D_a^{\geq p(a)}).$$

Iterating 1.5.1, as in [BBD], 2.1.3, we conclude that $(D^{p\leq 0}, D^{p\geq 0})$ and $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ are t-structures on D. We'll denote them t^p and ${}^pt$.

1.5.2.2 When $p \equiv 0$ then these t-structures coincide with the initial one.

1.5.3 PROPOSITION. *If p does not increase then both above t-structures are compatible with the filtration.*

1.5.3.1 LEMMA. *An arbitrary t-strucure on D is compatible with the filtration iff all k_a^* are t-exact* ∎

Proof of 1.5.3. Let us prove for example that t^p is compatible with the filtration. We have to prove that k_a^* is t^p-exact. By definition, $k_a^*(D^{p\leq 0}) \subset D^{p\leq 0}$. Let $x \in D_{bc}^{p\leq 0}, b \leq c$. Let us prove that $k_a^* x \in D_a^{p\leq 0}$ by induction on $c - b$. First , $D_{bc}^{p\leq 0} = \bigcap_{b\leq a\leq c} k_a^{!-1}\mathcal{D}_a \subset k_a^{!-1} D_{bc}^{\leq p(c)} = D_{bc}^{\geq p(c)}$ since p is non-increasing, and by 1.5.2.2. Hence, $k_c^* x \in D_{bc}^{\leq p(c)}$ since k_c^* is t-exact. Further, from exact triangle $j_{a-1!} j_{a-1}^! x \to w \to i_{a!} i_a^* x$ and induction assumption we deduce that $k_b^* x \in \mathcal{D}^{p(b)}$. Since $j^! \mathcal{D}^{p\geq 0} \subset \mathcal{D}^{p\geq 0}$. Hence $k_a^*(\mathcal{D}^{p\geq 0}) \subset \mathcal{D}^{p\geq 0}$ ∎

1.5.4 If $q : \mathbf{Z} \to \mathbf{Z}$ is another function then $(\mathcal{D}^p)^q = \mathcal{D}^{p+q}, \ldots$.

2. MIXED t-CATEGORIES AND DUALITY

2.1 Mixed t-categories

DEFINITION. a. *A mixed abelian category* is a bounded filtered abelian category M such that each M_a is semisimple.

b. *A mixed t-category* is an ft-bounded ft-category D such that the following *wedge axiom* holds:

(W) For each $a, b \in \mathbf{Z}, x \in M_a, y \in M_b$ one has $\mathrm{Hom}_D^i(x, y) = 0$ for $i > a - b$ (Here $M = D^0$ is core of D).

c. *A pure functor* between mixed categories is an f-exact functor (case a) or ft-exact functor (case b).

We will call a mixed t-category *finite* if it satisfies (F) (1.3.4), and *of finite type* if $(F)'$ holds.

2.1.1 LEMMA. *a. The core of a mixed* t*-category is a mixed abelian category.*

b. The bounded derived category of a mixed abelian category is a mixed t*-category.*

c. Let $D_1 \to D \to D_2$ *be an* ft*-exact sequence and* D *is mixed. Then* D_1, D_2 *are mixed.* ∎

2.1.2 REMARKS. (i) The definition a, c are taken from [BMS], which contains the reasons for using the term «mixed».

(ii) (M) is equivalent to the property ${}^V D^{\geq 0} \subset {}^{\perp} D^{V<0}$.

(iii) Compare this with [E], 0.1, where opposite inequalities are considered.

2.2 Cohomological amplitudes

Let D be an ft-category satisfying (F) (1.3.4), $a \in \mathbf{Z}$. Consider the canonical ladder (1.5.2):

$$D_{\leq a} \begin{array}{c} \xleftarrow{j^?} \\ \xrightarrow{j_!=j_?} \\ \xleftarrow{j^*=j^!} \\ \xrightarrow{j_*} \end{array} D \begin{array}{c} \xleftarrow{i_?} \\ \xrightarrow{i^*=i^?} \\ \xleftarrow{i_!=i_*} \\ \xrightarrow{i^!} \end{array} D_{\geq a+1} \qquad (2.2.0)$$

On $D_{\leq a}, D_{>a}$ we consider the induced t-structures. Then

$$\begin{array}{lll} (2.2.1) & i_!, j_! & \text{are } t\text{-exact by definition,} \\ & i^*, j^* & \text{are } t\text{-exact;} \\ (2.2.2) & j_*, i^! & \text{are left } t\text{-exact,} \\ & j^?, i_? & \text{are right } t\text{-exact.} \end{array}$$

Let us prove, for example, that $j_*(D^{\geq 0}_{\leq a}) \subset D^{\geq 0}$.

By [BDD], 1.3.4 (b) $D^{\geq 0} = D^{<0\perp}$; so let $x \in D^{<0}, y \in D^{\geq 0}_{\leq a}$. $\mathrm{Hom}(x, j^* y) = \mathrm{Hom}(j^* x, y) = 0$ by 2.2.1, and we are done.

2.2.3 LEMMA. *If* D *is a finite mixed* t*-category, then for* $b \leq a < c$ *one has*

(a) $j_*(D^{\leq 0}_{b,a}) \subset D^{\leq a-b}_{\geq b}$;

(b) $i_?(D^{\geq 0}_{a+1,c}) \subset D^{\geq a+1-c}_{\leq c}$

(c) $j^?(D^{\geq 0}_{\leq c}) \subset D^{\geq a-c}_{\leq a}$

(d) $i^!(D^{\geq 0}_{\geq b}) \subset D^{\leq a-b+1}_{\geq a+1}$

The proof reduces to the case $b = a-1, c = a+1$, where the easy argument similar to 1.3.4.4 does the job (note that $c \& d$ follow from $a \& b$ and exact triangles $i_? i^? x \to x \to j_? j^? x, i_! i^! x \to x \to j_* j^* x$). ■

2.3 PROPOSITION. *Let D be a finite mixed t-category, $p : \mathbb{Z} \to \mathbb{Z}$ be a function such that $p(a+1) \geq p(a)+1$. Then the t-structure t^p (cf. 1.5.2) is compatible with the filtration ($D^V_{\leq a}, D^V_{\geq a}$ (1.3.5), and p_t – with (${}^V D_{\leq a}, {}^V D_{\geq a}$).*

Proof. Let us prove, for example, the first assertion. By 1.4.1 (d) we have to prove that (in notation 1.5.2.1) $i_{a!} i^!_a$ are t^p-exact. They are left t^p-exact by definition; so it remains to verify that $i_{a!} i^!_a D^{p\leq 0} \subset D^{p\leq 0}$. Every $x \in D^{p\leq 0}$ is an extension of $x_b \in D_b^{\leq p(b)}$. By 2.2.3 (d) $i^!_a x_b \in D^{\leq p(b)+a-b} \subset D^{\leq p(a)}$ (for $b \geq a$; for $b < a$ one has $i^!_a x_b = 0$). It follows that $i^!_a x \in D^{p\leq 0}_{\geq a}$. ■

2.4 THEOREM-DEFINITION. *Let D be a finite mixed t-category. Put*

$$t^V = (D^{V\leq 0}, D^{V\geq 0}) := (D^{p\leq 0}, D^{p\geq 0});$$

$${}^v t = ({}^V D^{\leq 0}, {}^V D^{\geq 0}) := ({}^p D^{\leq 0}, {}^p D^{\geq 0})$$

where $p(a) = a, a \in \mathbb{Z}$. Then

$$(D^{V\leq 0}, D^{V\geq 0}; D^V_{\leq a}, D^V_{\geq a})$$

and

$$({}^V D^{\leq 0}, {}^V D^{\geq 0}; {}^V D_{\leq a}, {}^V D_{\geq a})$$

define on D two new structures of a mixed t-category. We'll denote them D^V and ${}^V D$ and call right and left Koszul dual to the initial mixed structure.

D^V (resp., ${}^V D$) satisfies (FL) (resp., (FR)) and we have

$${}^V(D^V) = {}^V(D)^V = D.$$

If M is a mixed abelian category, we put $M^V = D^b(M)^{V0}$; ${}^V M = ({}^V D^b(M))^0$.

Proof. By 2.3 it remains to prove the wedge axiom, which follows from 2.2.3 (a), (b). ■

2.5 REMARKS

(i) For an arbitrary p put $\bar{p}(a) = p(-a) + a$; $\bar{\bar{p}} = p$. Then $t^p = {}^{\bar{p}}(t^v)$; ${}^p t = ({}^v t)^{\bar{p}}$. One has $p(a+1) \geq p(a)+1$ iff $\bar{p}$ is non-increasing, cf. 2.3 and 1.5.3.

(ii) t-structures on $D_a = D^v_{-a}$ induced by t and t^v differ by a shift on a, and analogously for ${}^v D$.

(iii) One has $(D^{\mathrm{opp}})^v = ({}^v D)^{\mathrm{opp}}, (D(a)[b])^v = D^v(-a)[b-a]$.

2.6 PROPOSITION. *Let $D_1 \xrightarrow{i} D \xrightarrow{\pi} D_2$ be an ft-exact sequence of finite mixed categories. Then it may be embedded in a canonical two-step ladder*

$$\begin{array}{ccccc} & \overset{v_i}{\longleftarrow} & & \overset{v_\pi}{\longleftarrow} & \\ D_1 & \overset{i}{\longrightarrow} & D & \overset{\pi}{\longrightarrow} & D_2 \\ & \overset{v_i}{\longleftarrow} & & \overset{\pi_v}{\longleftarrow} & \end{array}$$

The exact sequences $\;{}^vD_2 \xrightarrow{v_\pi} {}^vD \xrightarrow{v_i} {}^vD_1; \quad D_2^v \xrightarrow{\pi_v} D^v \xrightarrow{i^v} D_1^v$ *are ft-exact.*

Sketch of the proof. Note that an arbitrary f-exact functor $C \to C'$ is uniquely determined by its restrictions on all C_a. So the construct, for example, π_v it suffices to know its values on $\mathrm{Irr}(D_a^v)$. It is not difficult to see that for $L \in \mathrm{Irr}(D_a^v)\, \pi_v L := j_{a*}L$ is good (cf. 1.3.6) ■

2.7 Now suppose that D is a mixed t-category of finite type.

2.7.1 For $x \in D$ put $\mathrm{supp}\, x = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} | gr_a H^b(x) \neq 0\}$
For $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ put

$$\overrightarrow{T}_{a,b} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid j \geq b; j - b \leq i - a\};$$

$$\overleftarrow{T}_{a,b} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid j \leq b; b - j \leq a - i\}.$$

2.7.2 DEFINITION. $\varinjlim D$ (resp., $\varprojlim D$) is the category whose objects are sequences $x = \{x_a \in D_{\leq a}\}_{a \in Z}$ (resp., $\{x_a \in D_{\geq a}\}_{a \in Z}$) together with transition maps φ_{ab} : $x_a \to x_b, a \leq b$, such that $\varphi_{bc}\varphi_{ab} = \varphi_{ac}$, inducing isomorphisms $x_a \tilde{\to} w_{\leq a} x_b$ (resp., $w_{\geq b} x_a \tilde{\to} x_b$), and such that there exists $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ (depending on x) such that for all a supp $x_a \subset \overrightarrow{T}_{pq}$ (resp., $\mathrm{supp}\, x_a \subset \overleftarrow{T}_{pq}$). A morphism $f : x \to y$ is a set of maps $f_a : x_a \to y_a$ such that $f_b \varphi_{ab} = \varphi_{ab} f_a$.

$\varprojlim D$ and $\varinjlim D$ satisfy all the axioms of a mixed t-category, except for boundedness of t-structure. For such a category D set D^f to be the full subcategory of x with finite support. ■

2.7.3 By 1.3.4.1 and wedge axiom one has canonical diagrams

$$\begin{array}{ccccc} & \overset{j_{a!}}{\longrightarrow} & & \overset{i_a^*}{\longrightarrow} & \\ D_{\leq a} & \overset{j_a^*}{\longleftarrow} & \varinjlim D & \overset{i_{a*}}{\longleftarrow} & (\lim D)_{\geq a} \\ & \underset{j_{a*}}{\longrightarrow} & & \underset{i_a^!}{\longrightarrow} & \end{array}$$

$$
\begin{array}{ccccc}
 & \overset{j_a^?}{\longleftarrow} & & \overset{i_{a?}}{\longleftarrow} & \\
(\varprojlim D)_{\leq a} & \overset{j_{a?}}{\longrightarrow} & \varprojlim D & \overset{i_a^*}{\longrightarrow} & D_{\geq a} \\
 & \overset{j_a^*}{\longleftarrow} & & \overset{i_{a*}}{\longrightarrow} &
\end{array}
$$

Define dual filtrations:

$$
{}^v(\varinjlim D)_{\leq a} = i_{a*}(\varinjlim D)_{\geq -a}, \quad {}^v(\varinjlim D)_{\geq a} = j_{-a*}D_{\leq -a};
$$

$$
(\varprojlim D)^v_{\leq a} = i_{-a?}D_{\geq -a}, \quad (\varprojlim D)^v_{\geq a} = j_{-a?}D_{\leq -a}.
$$

Construction of (2.4) yields new t-structures: t^v on $\varinjlim D$ and ${}^v t$ on $\varprojlim D$; we get (unbounded) mixed t-categories ${}^v \varinjlim D$, $\varprojlim D^v$.

Finally, put

$$
D^v = (\varinjlim D^v)^f; \quad {}^v D = ({}^v\varprojlim D)^f
$$

These are mixed t-categories of finite type which we call right and left Koszul dual to D.

2.7.4 We have ${}^v(D^v) = ({}^v D)^v = D$.

2.8 Let M be an abelian mixed category all whose objects have finite length, $I = \operatorname{Irr} M$. For $\alpha, \beta \in I$ put $P(M)_{\alpha,\beta} := \sum_i (-1)^i rk \operatorname{Ext}^i(\alpha, \beta)$, where rk is the length of an $(\operatorname{End}\beta, \operatorname{End}\alpha)$-bimodule. (Variant: if M is k-category then $rk = \dim_k$). So we get an $I \times I$-matrix $P(M)$.

2.8.1. LEMMA. $P({}^v M)P(M) = P(M)P(M^v) = 1_I$ ■

3. DG-ALGEBRAS LANGUAGE

In the remaining part of the text we fix a base field k; all the vector spaces and categories are supposed to be k-linear.

3.1 Let S be a set. *An S-algebra A* consists of finite dimensional vector spaces $A_{s_1 s_2}$ given for all $s_1, s_2 \in S$ together with associative multiplication $A_{s_1 s_2} \times A_{s_2 s_3} \to A_{s_1 s_3}$; we also assume that A is unitary, i.e. the algebras $A_{s,s}$ have units 1_s and $1_{s_1} f = f 1_{s_2} = f$ for $f \in A_{s_1 s_2}$. Note that if S is finite, then S-algebra is the same as usual unitary algebra A together with decomposition $1_A = \sum_{s \in S} 1_s$, where $1_s^2 = 1_s, 1_{s_1} 1_{s_j} = 0$ if $s_1 \neq s_j$ (just put $A_{s_1 s_2} = 1_{s_1} A 1_{s_2}$, $A = \oplus A_{s_1 s_2}$). Similarly one defines differential graded S-algebras(DGS-algebras); as usually we will consider any S-algebra as DGS-algebra (placed in degree 0), and for a DGS-algebra $A^\cdot$ one has S-algebra $H^0(A^\cdot)$.

Assume that we are given a function $w : S \to \mathbb{Z}$ (called weight); put $S_i = w^{-1}(i)$. It defines the grading on a DGS-algebra $A^\cdot$: put $A^\cdot_i := \bigoplus_{\substack{s_1,s_2 \in S \\ w(s_1)-w(s_2)=i}} A^\cdot_{s_1 s_2}$; $A_{ab} = \bigoplus_{\substack{w(s_1)=a \\ w(s_2)=b}} A_{s_1,s_2}$. We will say that $A^\cdot$ is an f-algebra if $A^\cdot_i = 0$ for $i > 0$; that $A^\cdot$ is an *ft-algebra* if, in addition, $A^j_\cdot = 0$ for $j > 0$; that $A^\cdot$ is *diagonal* if $H^j(A^\cdot) = 0$ for $j \neq 0$.

3.1.1 DEFINITION. (i) A DGS-algebra $A^\cdot$ is *0-semisimple* if it is of ft- type, $A^\cdot_0 = H^0(A^\cdot_0) = \oplus A_{ss}$, and each A_{ss} is semisimple algebra.

(ii) A DGS-algebra is *mixed* if it is 0-semisimple and $A^j_i = 0$ for $j < i$. ■

Hence an S-algebra A is mixed iff $A_i = 0$ for $i > 0$ and $A_0 = \oplus A_{ss}$ is semisimple.

3.1.2. REMARK. Say that $A^\cdot$ is *0-simple* if it is 0-semisimple and each A_{ss} is simple algebra. Since in 3.1.1. and below it is weights, and not S-grading, which are important, we can consider each 0-semisimple S-algebra as 0-simple I-algebra; where $I = \sqcup \mathrm{Irr}\, A_{ss}$.

If A is an S-algebra, then *(a left) A-module* X is a set $\{X_s\}_{s \in S}$ of vector spaces together with associative multiplications $A_{s_1 s_2} \otimes X_{s_2} \to X_{s_1}, 1_s \to \mathrm{id}_{X_s}$ (if S is finite this is a usual A-module X : put $X_s = 1_s X$ etc.). Similarly, for a DGS-algebra $A^\cdot$ we have have DG-modules X. In what follows we will assume that our A- and$A^\cdot$-modules are finite dimensional (i.e. $X^\cdot_s$ are zero for all but finitely many $s \in S, \cdot \in Z$, and each $X^\cdot_s$ is finite dimensional). Denote the corresponding categories $M(A), C(A^\cdot)$ respectively. If $A^\cdot$ is f-algebra, $C(A^\cdot)$ contains full subcategories $C(A^\cdot)_{\leq a} = \{X^\cdot : X^\cdot_s = 0 \text{ if } w(s) > a\}, C(A^\cdot)_{\geq a} = \{X^\cdot : X^\cdot_s = 0 \text{ if } w(s) < a\}$.

For $M \in C(A^\cdot)$ we put $M_a = \bigoplus_{w(s)=a} M_s$. Similarly categories of right A-modules $M(A)^r, C(A)^r$ etc. are defined.

3.1.3 EXAMPLE. Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a Z-graded algebra.It defines a Z-algebra $\tilde{A}$ with $\tilde{A}_{ij} = A_{i-j}$. $\tilde{A}$-modules are the same as ordinary graded A-modules.

3.2. Derived categories

3.2.1 LEMMA-DEFINITION *(cf. [I], ch. VI, n. 10). Let $A^\cdot$ be a DGS-algebra. Call a map $f : M \to N$ in $C(A^\cdot)$ a quasi-isomorphism if all $f^\cdot_s : M^\cdot_s \to N^\cdot_s$ are quasiisomorphisms.*

Let $D(A)$ be the localization of $C(A)$ with respect to the set of all quasiisomorphisms. This is a triangulated category. If $A^\cdot$ if an f-algebra, then localising $C(A)_{\leq a}$, $C(A)_{\geq a}$, one gets the categories $D(A)_{\leq a}, D(A)_{\geq a}$, which define the f-structure on $D\,(A)$. If A is an ft-algebra, then, in addition, $D(A)^{\leq 0} = \{X^\cdot : H^i(X^\cdot) = 0$ for $i > 0\}$, $D(A)^{\geq 0} = \{X^\cdot : H^i(X^\cdot) = 0$ for $i < 0\}$, define the ft-structure on $D(A)$ with core $M(H^0(A^\cdot))$. It is ft-bounded and satisfies (F') (1.3.4). If $A^\cdot$ is a mixed DG-algebra, then $D\,(A)$ is mixed category. ■

3.2.2 Functoriality

Let S, T be sets, $A^{\cdot}$(resp.$B^{\cdot}$) be a DG S(resp.T)-algebra. If $f : S \to T$ is a map, then an f-morphism $F : A \to B$ is just the collection of maps $F_{s_1 s_2} : A^{\cdot}_{s_1 s_2} \to B^{\cdot}_{f(s_1) f(s_2)}$ compatible with multiplications. It defines the exact functor $F^{\cdot} : C(B^{\cdot}) \to C(A^{\cdot}), F^{\cdot}(X^{\cdot})_s = F(1_s) X^{\cdot}_{f(s)}$, hence the exact functor $F^{\cdot} : D(B^{\cdot}) \to D(A^{\cdot})$. If $A^{\cdot}, B^{\cdot}$ are $f(ft)$-algebras and f preserves weights, then $F^{\cdot}$ is f (resp., ft)-exact.

3.2.3. Let $A^{\cdot}$ be 0-semisimple DG S-algebra, and $S_1 \subset S$ be a subset; put $S_2 = S - S_1$. It defines a Serre subcategory $M_{S_1} \subset M(:= M(H^0(A)))$ that consists of modules supported on S_1 (note that if $A^{\cdot}$ is 0-simple, see 3.1.1.1., then each Serre subcategory of M has this form), the corresponding thick ft-subcategory $D_{S_1} \subset D(:= D(A^{\cdot}))$ and the ft-quotient $D^{S_2} := D/D_{S_1}$ with core $M^{S_2} := M/M_{S_1}$.

Let A^{S_2} be the DG S_2-algebra $A^{S_2}_{s,t} = A_{s,t}$ for $s, t \in S_2, i : A^{S_2} \to A^{\cdot}$ be the embedding. We get the ft-exact functor $i^{\cdot} : D(A) \to D(A^{S_2})$.

3.2.3.1 LEMMA. *$i^{\cdot}$ induces an ft-equivalence $D^{S_2} \tilde{\to} D(A^{S_2})$.*

The proof goes along the same lines as in 2.6. ■

3.2.3.2 NOTATION. For $a, b \in Z$ put $S(a, b) = \{s \in S | a \le w(s) \le b\} \subset S$; $C_{ab}(A) = C(A)_{S(a,b)}$, $D_{ab}(A) = D(A)_{S(a,b)}$, $M(H^0(A))_{ab} = M(H^0(A))_{S(a,b)}$.

3.3 *From categories to algebras.*

3.3.1 LEMMA. *Let A be a semisimple algebra, S a set of isomorphism classes of simple left A-modules; for $L_s \in S$ put $D_s = \mathrm{End}_A(L_s)$; let M be a right A-module. Then the natural map*

$$M \to \bigoplus_{s \in S} \mathrm{Hom}_{D_s}(L_s, M \otimes_A L_s)$$

is an isomorphism. ■

3.3.2. Let M be a semisimple abelian k-category. Say that M is *k-finite* if $\mathrm{Irr}\, M$ is finite and for any $\overline{\overline{x}} \in M$ one has $\dim_k \mathrm{End}\, X < \infty$. Recall that a *fiber functor* is an exact functor $\varphi : M \to M(k)$ such that for every nonzero $\overline{\overline{x}} \in M, \varphi(\overline{\overline{x}}) \neq 0$. Evidently, such a functor always exists, the algebra $A_\varphi := \mathrm{End}\, \varphi$ is semisimple, and, by 3.3.1 a natural functor

$$M \to M(A_\varphi), \ M \mapsto \varphi(M)$$

is an equivalence of categories.

Let D be a finite semisimple t-category. Call *a fiber functor for D* a fiber functor ϕ for its core $M = D^0$. It defines a

$$\overline{\phi} = \bigoplus_i \phi^i : D \to GrM(k)$$

$\phi^i = \phi \circ H^i$. Put $\phi_i = \phi^{-1}$.

3.3.3 Now suppose that D is an ft-category with all D_a finite semisimple, and that for every a a fiber functor ϕ_a for D_a is given. We get the functors $\phi_a^i : D \to M(k), \phi_a^i(\bar{\bar{X}}) := \phi_a^i(Gr_a\bar{\bar{X}})$.

3.3.3.1 PROPOSITION. *Suppose that $D = D(A)$ for a 0-semisimple DG-algebra A. Consider the natural fiber functors ϕ_a := (underlying vector space) for $M(A_{aa})$. Then for any a, b, i, j the obvious maps*

$$H^i(A_{ab}) \to \mathrm{Hom}(\phi_b^j, \phi_a^{j+i}), \tag{3.3.3.2}$$

are isomorphisms (cf. [BMS], 2.1.2).

Proof. Let us construct an inverse map. Consider an object $M \in C^{\cdot}(A)$ with $M_a = A_{ab} \otimes_{A_{bb}} M_b (a \leq b)$ for some fixed $M_b \in M(A_{bb})[-j]$ that contains with non-zero multiplicity a representative from every $s \in \mathrm{Irr}\, A_{bb}$.Put $D = \mathrm{End}_{A_{bb}}(M_b)$. From 3.3.1. follows that

$$H^i(A_{ab}) \xrightarrow{\sim} \mathrm{Hom}_D(\phi_b^j(M_b), \phi_a^{i+j}(M_a)),$$

so, given a natural transformation $\phi_b^j \to \phi_a^{i+j}$, we get an element of $H^i(A_{ab})$. This defines an inverse map to (*). ∎

3.3.4 COROLLARY. *In the situation 3.2.2 assume that $f : S \xrightarrow{\sim} T, F_0 : A_0 \to B_0$ are isomorphisms, and A, B are 0-semisimple. Then $F^{\cdot}$ is equivalence iff F is quasiisomorphism.*

In particular, if A is an ft-algebra, and $\epsilon : A \to H^0(A)$ a natural projection, $\epsilon^{\cdot}$ is an equivalence iff $A^{\cdot}$ is diagonal. ∎

3.3.5 QUESTION. Whether any ft-category with semisimple D_a's comes from certain ft- algebra?

3.4 Standard functors

3.4.0 Let A be a 0-semisimple algebra with $A_s^j = 0$ for $j \neq 0$, $s \in S$. Fix some $a, b, c \in Z$, $a \le b < c$.

One has evident restriction functors $M_{ab}(A) \overset{j^*}{\leftarrow} M_{ac}(A)$, $M_{ac}(A) \overset{i^*}{\rightarrow} M_{b+1,c}(A)$.

3.4.1 This diagram may be uniquely completed to a diagram

$$M_{ab}(A) \quad \begin{matrix} \overset{j^?}{\leftarrow} \\ \overset{j_! = j_?}{\rightarrow} \\ \overset{j^* = j^!}{\leftarrow} \\ \overset{j_*}{\rightarrow} \end{matrix} \quad M_{ac}(A) \quad \begin{matrix} \overset{i_?}{\leftarrow} \\ \overset{i^* = i^?}{\rightarrow} \\ \overset{i_! = i_*}{\leftarrow} \\ \overset{i^!}{\rightarrow} \end{matrix} \quad M_{b+1,c}(A)$$

with an upper arrow left adjoint to a (neighbour) lower one.

Explicitly:

$$(j_! M)_p = \begin{cases} M_p & \text{for } p \in [a,b], \\ 0 & \text{otherwise} \end{cases}; \quad (i_! M)_p = \begin{cases} M_p & \text{for } p \in [b+1,c], \\ 0 & \text{otherwise} \end{cases}$$

$$(j^? M)_p = \operatorname{Coker} \Big(\bigoplus_{b+1 \le q \le c} A_{pq} \otimes A_{qq} M_p \to M_p \Big)$$

$$(i^! M)_p = \ker \Big(M_p \to \bigoplus_{a \le q \le b} \operatorname{Hom}_{A_{qq}} (A_{qp}, M_q) \Big)$$

$$(j_* M)_p = \begin{cases} M_p & \text{for } p \in [a,b], \\ \ker\Big(\bigoplus_{q \in [a,b]} \operatorname{Hom}_{A_{qq}}, (A_{qp}, M_q) \rightrightarrows \\ \rightrightarrows \bigoplus_{a \le q_1 < q_2 \le b} \operatorname{Hom}_{A_{q_1 q_1}} (A_{q_1 q_2} \otimes_{A_{q_1 q_2}} A_{q_2 p}, M_{q_1}) \Big) & \text{for } p \ge b+1 \end{cases}$$

$$(j_? M)_p = \begin{cases} \operatorname{Coker}\Big(\bigoplus_{b+1 \le q_1 < q_2 \le c} A_{pq_1} \otimes_{A_{q_1 q_1}} A_{q_1 q_2} \otimes_{A_{q_2 q_2}} M_{q_2} \rightrightarrows \\ \rightrightarrows \bigoplus_{b+1 \le q \le c} A_{pq} \otimes_{A_{qq}} M_q \Big) & \text{for } p \in [a,b], \\ M_p & \text{for } p \ge b+1 \end{cases}$$

3.4.2 Exactness

$i^*, j^*, i_!, j_!$ are exact; $j^?, i_?$ are left exact, $j_*, i^!$ right exact.
If $a = b$ (resp. $b+1 = c$) then j_* (resp. $i_?$) is exact.

3.4.3 One has canonical exact sequences with adjunction maps as arrows:

(a) $0 \to j_! j^! M \to M \to i_* i^* M \to 0$

(b) $0 \to i_! i^! M \to M \to j_* j^* M$

$(b)^0$ $i_? i^? M \to M \to j_? j^? M \to 0$

(cf. [BBD], 1.4.1).

3.5 Bar-construction

In this n° we'll construct certain canonical representatives for the derived of functors from 3.4. We assume the conditions 3.4.0 to be satisfied.

3.5.1 Let $a, c \in \mathbf{Z}$; $a \leq c$, $M \in M_{ac}(A)$, $N \in M_{ac}(A)^r$.
Define a complex

$$B_\cdot(N, M)_{ab} \in C^{a-c,0}(k) \qquad \text{as follows.}$$

Put

(3.5.1.1)
$$B_i(N, M)_{ac} = \bigoplus_{a \leq p_0 < p_1 < \ldots < p_i \leq c} N_{p_0} \otimes_{A_{p_0 p_0}} A_{p_0 p_1} \otimes_{A_{p_1 p_1}} \cdots \cdots \otimes_{A_{p_{i-1} p_{i-1}}} A_{p_{i-1} p_i} \otimes_{A_{p_i p_i}} M_{p_i}$$

Let $\partial_j : B_i(N, M)_{ac} \to B_{i-1}(N, M)_{ac}$ be induced by, respectively, $N_{p_0} \otimes A_{p_0 p_1} \to N_{p_1}$ for $j = 0$, $A_{p_j p_{j+1}} \otimes A_{p_{j+1} p_{j+2}} \to A_{p_j p_{j+2}}$ for $1 \leq j < i-1$, $A_{p_{i-1} p_i} \otimes M_{p_i} \to M_{p_{i-1}}$ for $j = i-1$; put $d = \sum (-1)^i \partial_j : B_i \to B_{i-1}$.

3.5.2 Dually, for $M, N \in M_{ac}(A)$, define $B^\cdot(N, M)_{ac} \in C^{0, c-a}(k)$ with components

$$B^i(N, M) = \bigoplus_{a \leq p_0 < \ldots < p_i \leq c} \operatorname{Hom}_{A_{p_0 p_0}}(A_{p_0 p_1} \otimes A_{p_1 p_1} \ldots \otimes A_{p_{i-1} p_i} \otimes_{A_{p_i p_i}} N_{p_i}, M_{p_0})$$

We have

(3.5.2.2)
$$H^i B^\cdot(N, M) = \operatorname{Ext}^i_{M_{ac}(A)}(N, M)$$

3.5.3 More generally, for $a \leq b \leq c$, let

$$B_\cdot(N, M)_{(a;b,c)} \subset B_\cdot(N, M)_{ac}$$

(resp., $B^\cdot(N, M)_{(a,b;c)} \subset B^\cdot(N, M)_{ac}$) be the subcomplex with components of the same form as (3.5.1.1) (resp., (3.5.2.1)) but with the summation extended over all $a \leq p_0 < \ldots < p_i \leq c$, with $p_1 \geq b$ (resp., $p_{i-1} \leq b$).

3.5.4 Let $\bar{A}_{pp}$ denote the A-bimodule with $(\bar{A}_{pp})_p = A_{pp'}$ and $(\bar{A}_{pp})_i = 0$ for $i \neq p$.

LEMMA. *Let $M \in M_{ac}(A)$. The left derived of $j^?$ (resp., right derived of $i^!$) may be represented by the complex $Lj^? M \in C^{b-c,0}(A)$ with components $(Lj^? M)_p = B_\cdot(\bar{A}_{pp}, M)_{(a;b+1,c)}$, $a \leq p \leq b$; and multiplication*

$$A_p \otimes (Lj^? M)_q \to (Lj^? M)_p$$

induced by $A_{pq} \otimes A_{qp_1} \to A_{pp_1}$ (resp., $Ri^! M \in C^{0,b-a+1}(A)$; $(Ri^! M)_p = B^\cdot(\bar{A}_{pp}, M)_{a,b;c}$, $b+1 \leq p \leq c$; and multiplication induced by

$$A_{pq} \otimes \operatorname{Hom}(A_{p_0 p_1} \otimes \ldots \otimes A_{p_{i-1} q}, M_{p_0}) \to \operatorname{Hom}(A_{p_0 p_1} \otimes \ldots \otimes A_{p_{i-1} p}, M_{p_0})).$$

∎

EXERCISE. Define $R_{j_*}M \in C^{0,b-a}(A)$; $Li_? M \in C^{b-c+1,0}(A)$ for $M \in M(A_{ab})$ (resp., $M \in M(A_{b+1,c})$).

3.5.5 More generally, let A be an arbitrary 0-semisimple DG-algebra. For $N, M \in C^\cdot(A)$ the formulas of 3.5.1- 3.5.3 define some double (or triple) complexes; let $B_\cdot(N,M), \ldots$ etc. be the corresponding simple complexes.

The construction 3.5.4 defines functors $j^?, i^!$ between derived categories. If A is mixed, then the same estimation on the length of complexes as in 3.5.4 holds, so we get amplitudes 2.2.3!

3.6 Koszul duality

Let A be a mixed DGS-algebra.

3.6.1 Define a new DGS-algebra $\check{A}$ as follows. First, the weight function $\check{w} : S \to Z$ for it will be $\check{w} := -w$.

Further, put $\check{A}_{ss} = A_{ss}$. For $s_1, s_2, w(s_1) = a < b = w(s_2)$ put

$$\check{A}_{s_2,s_1} = B^\cdot(\bar{A}_{s_1 s_1}, \bar{A}_{s_2 s_2})_{ab}[b-a+1]$$

where $\bar{A}_{ss}$ denotes an A-bimodule with $(\bar{A}_{ss})_s = A_{ss}, (\bar{A}_{ss})_t = 0$ for $t \neq s$. $\check{A}_{s_2,s_1}$ has natural left $A_{s_2 s_2}$ and right $A_{s_1 s_1}$-module structures. Next note that $B^\cdot(\bar{A}_{s_1 s_1}, \bar{A}_{s_2 s_2})$ is nothing but a simple complex associated with $a(b-a-1)$-fold complex of length 1 in each dimension ($(b-a-1)$-dimensional cube). This structure induces natural pairings $\check{A}_{s_1 s_2} \otimes \check{A}_{s_2 s_3} \to \check{A}_{s_1 s_3}$.

So one gets a mixed DG-algebra $\check{A}$ which is called *Koszul dual* to A.

3.6.2 LEMMA. *One has canonical-quasiisomorphism* $A \to \check{\check{A}}$. ■

We leave the proof to the reader (cf. [Mo]).

The case of neighbouring weight follows from

3.6.2.1 LEMMA. *Let* A_0, A_1 *be semisimple algebras,* M *— a left* A_0*- and right* A_1*-module (i.e. left* $A_0 \otimes A_1^{\mathrm{opp}}$*-module). Then one has canonical isomorphisms of left* $A_0^{\mathrm{opp}} \otimes A_1$*-modules*

$$\mathrm{Hom}_{A_0}(M, A_0) \cong \mathrm{Hom}_{A_1}(M, A_1) \cong \mathrm{Hom}_{A_0 \otimes A_1^{\mathrm{opp}}}(M, A_0 \otimes A_1^{\mathrm{opp}});$$

here in the first (resp., second, third) term we consider homomorphisms on left A_0*-modules (resp., right* A_1*-modules, left* $A_0 \otimes A_1^{\mathrm{opp}}$*-modules).*

If A_0, A_1 *are* k*-algebras over a field* k*, then these* $A_0^{\mathrm{opp}} \otimes A_1$*-modules are isomorphic to* $\mathrm{Hom}_k(M, K)$ *with the induced* $A_0^{\mathrm{opp}} \otimes A_1$*-module structure.* ■

3.6.3 Now define canonical functors

$$(*) \qquad D(A)_{-b,-a} \overset{\ell_A}{\leftarrow} D_{a,b}(A) \overset{r_A}{\to} D(A)_{-b,-a}$$

Namely, for $M \in C_{ab}(A)$, $a \le p \le b$, put

$$\left.\begin{array}{r} \ell_A(M)_p = Lj_p^? M[p-b] \\ r_A(M)_p = Ri_p^! M[p-a] \end{array}\right\} \in C^{\cdot}(k)$$

where $j_p : [pp] \hookrightarrow [pb]; i_p : [pp] \hookrightarrow [ap]$, and $Lj^?, Ri^!$ are as in 3.5.4, 3.5.5 $\check{A}$-module structures on $\ell_A(M), r_A(M)$ is defined in the same manner as the multiplication on $\check{A}$ in 3.6.1.

After derivation, one gets (*).

3.6.4 THEOREM*(i) One has canonical isomorphism of functors.*

$$\ell^{\check{A}} \circ r_A \simeq r_{\check{A}} \circ \ell_A \simeq \mathrm{Id}_{D(A)_{ab}}$$

(we identify $D(A)$ with $D(\check{\check{A}})$ using 3.6.2), so r_A, ℓ_A are equivalences.

(ii) The equivalence ℓ_A (resp., r_A) transform the canonical mixed structure on $D(A)_{-b,-a}$ into ${}^vD(A)_{ab}$ (resp., $D(A)_{ab}^v$. ∎

3.6.5 EXAMPLE. (S-Λ-duality,one variable). Let $A_{ab} = k$, $A_{ab} \otimes A_{bc} \to A_{ac}$ = identity. Then one has $\check{A}_{ab} = k$; $A_{ab} \otimes A_{bc} \to A_{ac}$ = identity if $a = b$ or $b = c$, and zero otherwise. $D(A) = D(\check{A})$ is just $DF(M(k))$; $M(A)$ = generalized filtered objects; $M(A) = C^{\cdot}(k)$ (cf. (1.4.3)).

3.7 Let A be a mixed DG-algebra. In notations 3.2.3 put $A_{S_2} := {}^v(A^{vs}) = (({}^vA)^{S_2})^v$. (here $A^v = {}^vA := \check{A}$).Then 3.2.3 and 2.6 imply that one has a canonical ft-equivalence $D(A_{S_2}) \tilde{\to} D(A_{S_2})$. Note that if A is diagonal then A_{S_2} need not be diagonal.

QUESTION. Whether $D(B)$ for an arbitrary mixed DG-algebra B is ft-equivalent to $D(A_{S_2})$ for some diagonal A?

All mixed t-categories we know come this way.

4. KOSZUL CATEGORIES AND ALGEBRAS

4.0 Let D be a bounded t-category with the core M. We will say that D is *of derived type* if Ext_D^i is generated by Ext^1, i.e. for any $x, y \in M$ and $\alpha \in \mathrm{Ext}_D^i(x,y), i > 1$, there exists a sequence $x = x_0 \overset{\alpha_1}{\to} x_1 \overset{\alpha_2}{\to} \dots \overset{\alpha_{i-1}}{\to} x_{i-1} \overset{\alpha_i}{\to} x_1 = y, x_j \in M, \alpha_j \in \mathrm{Ext}^1(x_{j-1}, x_j)$ such that $\alpha = \alpha_i \circ \dots \alpha_1$. Equivalently, this means that the cohomology bifunctor

$\mathrm{Ext}^{\cdot}_D$ on $M^0 \times M$ is effacable by either of variables, i.e. for any $x, y \in M, \alpha \in \mathrm{Ext}^i_D(x,y), i > 1$, there exists an injective morphism $\beta_0 : y \hookrightarrow z$ in M such that $\beta \circ \alpha = 0$. Note that for any t-category there is (unique) morphism $\mathrm{Ext}^{\cdot}_M \to \mathrm{Ext}^{\cdot}_D$ of cohomology bifunctors of $M^0 \times M$ (here $\mathrm{Ext}^{\cdot}_M := \mathrm{Ext}^{\cdot}_{D^b(M)}$), which is identity on M and D is of derived type iff this morphism is isomorphism. If one has a realisation functor $r : D^b(M) \to D$ (i.e. an exact functor which is identify on M), then D is of derived type iff r is equivalence of categories.

4.1 Let D be a mixed t-category with the core M. Say that

– D is of v-*derived* type if for any $a, b \in \mathbb{Z}$, $x \in M_a$, $y \in M_b$ one has $\mathrm{Ext}^i_D(x,y) = 0$ if $i \neq a - b$

– D is *Koszul* is it is both of derived and v-derived type.

4.1.1 LEMMA. *Assume that D is of finite type, hence* ${}^vD, D^v$ *are defined.*

a. *D is of derived type iff either of the following holds:*

(i) *The functors* $i_? : D_{\geq a} \to \lim\limits_{\leftarrow} D$ *are t-exact.*

(ii) *One has* ${}^vM_a[-a] \subset \lim\limits_{\leftarrow} M$.

Same with dual conditions for j_*, M^v.

b. *D is of v-derived type iff either* ${}^vD$ *or* D^v *are of derived type.*

c. *D is Koszul iff such is* ${}^vD$ *or* ${}^vD$.

d. *If D is of derived type, then D is Koszul iff any* $L \in M_a$ *admits in M a resolution* $\ldots P_1 \to P_0 \to L \to 0$ *such that* P_i *is projective covering of a weight* $a - i$ *object.* ∎

REMARKS. D is of v-derived type iff $(D^v_{\leq 0}, {}^vD_{\geq 0})$ is t-structure on D (cf. [E]).

From now on we will assume that our category came from a 0-simple DG S-algebra $A : D = D(A)$; for $s \in S$ let L_s be the corresponding simple object. Hence D is of derived type iff A is diagonal (in this case we will always assume that $A = H^0(A)$), and D is of v-derived type iff $\check{A}$ is diagonal. We will say that a graded algebra A is a *Koszul* one if $D(A)$ is Koszul category; by 4.1.1 the dual algebra $\check{A}$ is also a Koszul one: then

$$\check{A}_{s_1 s_2} = \mathrm{Ext}_A^{w(s_1)-w(s_2)}(L_{s_1}, L_{s_2}).$$

4.2 We will say that a mixed algebra B is *quadratic* if it is generated (over B_0) by B_{-1} with only relations in B_{-2}. So to define a quadratic algebra one should take a semisimple algebra B_0, a B_0-bimodule$V$, and a sub-B_0-bimodule$W \subset V \otimes V$. The corresponding algebra $B_{\cdot} = B_0[V; W]$ is the quotient of the tensor algebra $T(V)\cdot = T_{B_0}(V)\cdot, T(V)^i = V \otimes_{B_0} \ldots \otimes_{B_0} V$ (i times) by the ideal generated by $W \subset T(V)_2$ (hence $V = B_1, W = \mathrm{Ker}(B_1 \underset{B_0}{\otimes} B_1 \to B_{-2})$). Now 3.6.4 implies

4.2.1 LEMMA. *If $A^{\cdot}$ is diagonal algebra, then $H^0(\check{A})$ is quadratic. Explicitly, $H^0(\check{A}) = A_0[A^*_{-1}; A^*_{-2}]$, where $A^*_{-2} \hookrightarrow A^*_{-1} \underset{A_0}{\otimes} A^*_{-1}$ is dual to multiplicaition map.* ∎

In particular Koszul algebras are quadratic. Below we will see that the above definition of Koszul algebras essentially coincides with the one of Priddy [P], see also [Lö].

4.3 Clearly 4.2.1 implies that $A \mapsto A^! := H^0(\check{A}^{\cdot})$ is an involution on the category of quadratic algebras; call it naive duality. The Koszul algebras are just the quadratic ones for which naive dual coincides with Koszul dual.

REMARK. For an algebra A consider an $S \times S$-matrix $P(A)_{s_1 s_2} = rkA_{s_1 s_2}$, where rk is length of $A_{s_1 s_1} \times A_{s_2 s_2}$-bimodule. Then 2.8 implies that if A Koszul, then $P(A)$ $P(A^!) = 1_S$. We do not know whether this numerical Koszul property suffice for A to be Koszul.

4.4 For a quadratic algebra A put $K^i = K^i(A) := \mathrm{Hom}_{\text{right A-mod}}(A^! \cdot 1_{-i}, A \cdot 1_{-i})$ where $1_{-i} \in A_0$ is projector on weight $-i$ component. These K^i are $(A, A^!)$-bimodules. Define the differential $d : K^i \to K^{i+1}$ by formula $df(u) = \sum f(uv_i^*)v_i$; here $v_i \in A_{-1}, v_i^* \in A^!_{-1} = A_1^*$ are dual basises: $\sum v_i^* \otimes v_i = id \in \mathrm{Hom}_{\text{right} A_0\text{-mod}}$ $(A_1, A_1) \subset A_i^* \otimes A_i$. Clearly, d is morphism of bimodules and $d^2 = 0$; this is Koszul complex of A. Note that K, considered as A-module, has natural grading $K^{i,j} :=$ Hom $(1_{-i+j}A^! 1_{-i}, A1_{-i})$. Clearly $d : K^{i,j} \to K^{i+1,j+1}, K^{i,j} = 0$ for $j > 0, K^{i,0} = A \cdot 1_{-i}$. Hence $K^{i,*}$ is complex of projective A-modules, augmented by $K^{i,0} \to A_0 1_{-i}$.

4.4.1 LEMMA. *A is Koszul algebra iff $K^{\cdot,*}$ is resolution of A_0.*

Proof. Assume that A is Koszul algebra. Consider the resolution K^{i*} (4.1.1 d) of the module $A_0 \cdot 1_{-i}$. It is easy to see that it is exactly the Koszul complex defined above, hence K^{i*} is acyclic. Conversely, if Koszul complex is acyclic, then $K_s = K \cdot 1_s$ is resolution of L_s (4.1.1 d). ∎

4.4.2 Let A be a Koszul algebra, A be its dual. If A if finite dimensional, one has canonical equivalence of categories $D(\check{A}) \simeq D(A) \simeq D(\check{A})$ that correspond to identifications $D(\check{A}) \simeq {}^{v}D(A)$, $D(\check{A}) \simeq D(A)^v$ (in general case one should worry a bit about finiteness conditions for modules and complexes, see 2.7). To define this equivalence explicitly, consider the Koszul complex $K'^{\cdot}$ for A with reserved multiplication (each K'^i is $(\check{A}, A)$-bimodule, projective as A-module). It is easy to see that the equivalence $D(A) \to D(A)$ that corresponds to $D(A) = D(A)^v$ is given by formula $C^{\cdot} \mapsto K'^{\cdot} \underset{A}{\otimes} C^{\cdot}$; one defines the second equivalence similarly.

A particular case is BGG-duality [BGG_1] (see introduction); for other important examples see [K]. Note that this equivalence, as Deligne noticed, resembles Fourier transform: e.g. it transform projective modules («constant functions») to irreducible ones («δ-functions).

4.5 Let $A_{\cdot} = A_0[V, W]$ be a quadratic algebra, hence $A_{-n} = T(V)^n / \sum_{1 \leq i \leq n-1} W_i^n$, where $W_i^n := V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i-1} \subset T(V)^n$.

To formulate the next lemma we will need a bit of linear algebra terminology. An n-tuple (of linear subspaces) $U = (U; U_1, \ldots, U_n)$ consists of a linear space U and its subspaces $U_1, \ldots, U_n \subset U$. The n-tuples form a groupoid with direct sum operation: $(U; U_1, \ldots) \oplus (V; V_1, \ldots) = (U \oplus V, U_1 \oplus V_1, \ldots)$. Rank of n-tuple $(U; U_1, \ldots)$ is $\dim U$.

An t-tuple is called distributive if either of the following equivalent conditions hold

(i) it is isomorphic to a sum of rank 1 n-tuples

(ii) $U_1, \ldots, U_n$ generate a distributive lattice of subspaces in U.

Now we may formulate.

4.5.1 LEMMA. *$A_{\cdot}$ is Koszul algebra iff $\tau^n = (T(V)^n; W_1^n, \ldots, W_{n-1}^n)$ is distributive $(n-1)$-tuple for each n.*

Proof. A bit more about n-tuples. A n-tuple is indecomposable if it is not isomorphic to a sum of a non-trivial n-tuples. Say that a property of n-tuples is additive if it holds for the sum $U_1 \oplus U_2$ iff it holds for each U_1, U_2. Clearly, the distributivity property is additive. To verify that two additive properties are equivalent it suffices to do this for indecomposable n-tuples only.

EXAMPLE. A triple $(U; U_1, U_2, U_3)$ is distributive iff $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$.

(Since any indecomposable triple is either of rank 1 or is 3 different lines on a plane.)

Let $\mathcal{U} = (U; U_1, \ldots, U_n)$ be an n-tuple. For $-n-1 \leq i \leq 0$ put $K^i = K^i(\mathcal{U}) := U_1 \cap \ldots \cap U_{-i-1} / (U_1 \cap \ldots \cap U_{-i-1} \cap (U_{-i+1} + U_{-i+2} + \ldots + U_n)$; hence $K^{-n-1} = U_1 \cap \ldots \cap U_n, \ldots, K^{-1} = U/U_2 + \ldots + U_n, \mathcal{K}^0 = U/U_1 + \ldots + U_n$. The obvious maps $d : K^n \to K^{n+1}$ make $K^{\cdot}$ a complex; call it Koszul complex of $\mathcal{U}$. Say that $\mathcal{U}$ is acyclic if $K(\mathcal{U})^{\cdot}$ is acyclic complex. The example above shows that acyclicity is equivalent to the following condition:

(*) The triples $(U; U_1 \cap \ldots \cap U_1, U_{i+1}, U_{i+2} + \ldots + U_n), i = 1, \ldots, n-2$, are distributive.

Say that n-tuple $\mathcal{U} = (U; U_1, \ldots, U_n)$ is predistributive if $n-1$-tuples $\mathcal{U}_i := (U; U_1, \ldots, \hat{U}_i, \ldots U_n), i = 1, \ldots, n$, are distributive.

Clearly both acyclicity and predistribuitivity are additive properties.

4.5.2 LEMMA. *n-tuple is distributive iff it is both acyclic and predistributive.*

It is easy to see that 4.5.2 implies 4.5.1 Indeed, $K^{\cdot}(\tau^n)$ is just the degree n component of $K^{\cdot}(A)$. Hence A is Koszul algebra iff all τ^n's are acyclic. Now note that predistributivity of τ^n follows if one knows that τ^i's are distributive for $i < n$ (since $\tau^n_i = \tau^i \otimes \tau^{n-i}$ in obvious notations). So 4.5.2 proves 4.5.1 by induction by n.

It remains to prove 4.5.2. The «only if » is trivial; the only thing is that acyclicity& predistibuitivity imply distributivity.

The following notations will be convenient. Given a n-tuple $v = (V; V_1, \ldots, V_n)$, say that a subspace $A \subset V$ is splitting for V if there exists $B \subset V$ such that $A \oplus B = V, (A \cap V_i) + (B \cap V_i) = V_i$ for any i. Hence V is indecomposable iff it has no proper splitting subspaces. We will use the following easy facts. Let v be a n-tuple. Then

4.5.2.1 The subspace $V_1 \cap \ldots \cap V_i$, or $V_1 + \ldots + V_i$, is splitting for $\mathcal{V}$ iff it is splitting for $n - i$-tuple$(V; V_{i+1}, \ldots, V_n)$.

4.5.2.2 Assume that $(V_1 \cap \ldots \cap V_i) \cap (V_{i+1} + \ldots + V_j) = 0$ and $V_1 \cap \ldots \cap V_i$ is splitting for $n - j + 1$-tuple$(V; V_{i+1} + \ldots + V_j, V_{j+1}, \ldots, V_n)$. Then $V_1 \cap \ldots \cap V_i$ is splitting for V.

Now let us return to 4.5.2 So let $u = (U; U_1, \ldots, U_n)$ be a predistributive acyclic n-tuple which is not distributive. We may assume that 4.5.2 holds for i-tuples, $i < n$ (induction by n), and that u is indecomposable (additivity). Clearly $n \geq 4$ and U_i are proper subspaces of U. One also has

4.5.2.3 $U_1 \cap U_2 = 0, U_{n-1} + U_n = 0$ (Since $n - 1$-tuples$(U; U_1 \cap U_2, U_3, \ldots, U_n)$, $(U; U_1, \ldots, U_{n-2}, U_{n-1} + U_n)$ satisfy 4.5.2, they are distributive, hence $U_1 \cap U_2, U_{n-1} + U_n$ are splitting for u by 4.5.2.1)

Consider two cases:

$n = 4$. Then $U_1 \cap U_3 \cap U_4 = U_2 \cap U_3 \cap U_4 = 0$ (in fact, 4.5.2.1 implies that for an indecomposable $n-$tuple$v$ of rank $> i$ one has $V_1 + \ldots + V_{n-1} = V; V_1 \cap \ldots \cap V_{n-1} = 0$). Moreover, $(U_1 + U_2) \cap U_3 \cap U_4 = 0$ (since $(U_1 + U_2) \cap U_3 \cap U_4 = [(U_1 + U_2) \cap U_3] \cap [(U_1 + U_2) \cap U_4] = [(U_1 \cap U_3) + (U_2 \cap U_3)] \cap [(U_1 \cap U_4) + (U_2 \cap U_4)]$, by predistributivity, and the subspaces $U_1 \cap U_3, U_2 \cap U_3, U_1 \cap U_4, U_2 \cap U_4$ are linearly independent by the previous remark). Hence, $U_3 \cap U_4 = 0$ (since by 4.5.2.1 it is splitting subspaces for u), so $U = U_3 \oplus U_4$ decomposes u (predistributivity). Contradiction ∎

$n > 4$. Note that 4.5.2.3 implies that u remains acyclic after arbitrary transpositions of $U_1, \ldots, U_{n-2}$, so we may assume that for certain $1 \leq i \leq n - 3$ one has $A = U_1 \cap \ldots \cap U_i \neq 0$ and each $i + 1$-tuple from $U_1, \ldots, U_{n-2}$ intersects by zero. Put $B = U_{i+1} + \ldots + U_{n-2}$. Then $(U; A, B, U_{n-1}, U_n)$ is predistributive quadruple satisfying

4.5.2.3 (e.g. $A \cap B = (A \cap U_{i+1}) + \ldots + (A \cap U_{n-2}) = 0$ by predistributivity), hence acyclic. So, it is distributive, hence A is splitting for u by 4.5.2.2. Since $A \neq 0$ we get contradiction. ∎

5. FIRST EXAMPLES

In this n^0 we consider the most simple examples of mixed categories of geometric origin.

5.1 Perverse sheaves over a disk

Let M be an abelian category.

5.1.1 Consider the following abelian categories and functors:

$$(5.1.1.1) \qquad M(\eta) \; \begin{matrix} \xrightarrow{j_!} \\ \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{matrix} \; M(S) \; \begin{matrix} \xrightarrow{i^*} \\ \xleftarrow{i_*} \\ \xrightarrow{i^!} \end{matrix} \; M(s)$$

$M(s) = M$ (« sheaves over a closed point»)

$M(\eta)$ («sheaves over a generic point»): Objects: pairs $(M \in M, N \in \operatorname{End} M$ («logarithm of monodromy»); Morphism $(M, N) \to (M', N')$ is a map $f : M \to M'$ such that $fN = N'f$.

$M(S)$ (« sheaves over a disk»): Objects: $(M_s \in M(s); (M_2, N) \in M(\eta)$; $\varphi : M_s \to M^N := \ker N)$;

A morphism $(M_s; M_\eta, N : \varphi) \to (M'_s; M'_\eta, N' : \varphi')$ is a pair of maps $f_s : M_s \to M'_s; f_\eta : (M_{\eta'}, N) \to (M'_\eta, N') \in \operatorname{Mor} M(\eta)$ such that $f_\eta \varphi = \varphi' f_s$.

$j_!(M, N) = (0; M, N; 0)$ (exact),

$j^*(M_s; M_\eta, N; \eta) = (M_\eta, N)$ (exact),

$j_*(M; N) = (M^N; M, N; \mathrm{id}_{M^N})$ (left exact)

$i^*(M_s; M_\eta, N; \varphi) = M_s$ (exact)

$i_*(M_s) = (M_s; 0, 0; 0)$ (exact)

$j^!(M_s; M_\eta, N; \varphi) = \ker \varphi$ (left exact)

As usually, in (4.1.1.1) an upper arrow is left adjoint to neighbour lower one.

Let $D(s), \ldots$ etc. denote the corresponding bounded derived categories.

Then (4.1.1.1) induces the two-step ladder

$$(5.1.1.2) \qquad D(\eta) \; \begin{matrix} \xrightarrow{j_!} \\ \xleftarrow{j^*} \\ \xrightarrow[Rj_*]{} \end{matrix} \; D(S) \; \begin{matrix} \xrightarrow{i^*} \\ \xleftarrow{i_*} \\ \xrightarrow[Ri^!]{} \end{matrix} \; D(s)$$

5.1.1.3 We have $Ri^!(M_s^\cdot \to M_\eta^{\cdot N})$ = a simple complex associated with a double one:

$$(\underset{0}{M_s^\cdot} \to \underset{1}{M_\eta^\cdot} \xrightarrow{N} \underset{2}{M_\eta^\cdot})$$

5.1.2 An alternative description of $D(S)$.

5.1.2.1 Let $C(S)$ be a category of triples

$$(M_s^\cdot \in C^b(M_s); (M_\eta^\cdot, N) \in C^b(M); \varphi : M_s^\cdot \to K(N)),$$

where for a map of complexes $f : A \to B$ $K(f) := C(f)[-1]$. A map $f : M^\cdot \to M^{\cdot\prime}$ is a pair $f_1 : M_s^\cdot \to M_s^{\cdot\prime}; f_\eta : M_\eta^\cdot \to M_\eta^\prime$ such that $N' f_\eta = f_\eta N$; $K(f_\eta) = \varphi' f_s \cdot f$ is called a *quasiisomorphism* if so are f_s, f_η.

Put $\tilde{D}(S)$ to be the localization of $C(S)$ with respect to quasiisomorphisms. (It may be also obtained as usually in two steps by localization of the appropriate homotopy category).

The evident functor $C^b(M(S)) \to \tilde{C}(S)$ induces

$$D(S) \to \tilde{D}(S).$$

5.1.2.2 LEMMA. *The last functor is an equivalence.* ∎

5.1.2.3 In terms of $\tilde{D}(S), Rj_*$ is conveniently described. Namely, for $(M_\eta, N) \in M(\eta)$

$$Rj_*(M_\eta, N) = (\overset{M_\eta}{\overset{\uparrow}{M_\eta}};\ (M_\eta, N);\ \varphi = \mathrm{id}_{K(N)}) \in \tilde{D}(S).$$

5.1.3 Apply the glueing theorem 1.5.1 to the perversity $p(s) = 0, p(\eta) = -1$: i.e. put

$$D(S)^{v\le 0} = \{M_s^\cdot \to M_\eta^\cdot : M_\eta^\cdot \in D(\eta)^{\le -1};\ M_s^\cdot \in D(s)^{\le 0}\};$$

$$D(S)^{v\ge 0} = \{M_s^\cdot \to M_\eta^\cdot : M_\eta^\cdot \in D(\eta)^{\ge -1};\ M_s^\cdot \to M_\eta^\cdot \xrightarrow{N} M_\eta^\cdot \in D^{\ge 0}\}.$$

$M(S)^v := D(S)^v 0$ («perverse sheaves over a disk»)

5.1.3.1 LEMMA. $M(S)^v = \{M_s^\cdot \to M_\eta^\cdot : M_\eta^\cdot \in D(\eta)^{-1};\ K(\varphi) \in D(s)^0\}.$ ∎

5.1.4 For $M = (M_s^\cdot \to M_\eta^\cdot) \in M(S)^v$ put

$$\psi(M) = H^{-1}(M_\eta^\cdot) \quad (\text{«vanishing cycles»})$$

$$\phi(M) = H^0(K(\varphi)) \quad (\text{«nearby cycles»}).$$

We have natural maps

$$u : \psi(M) \to \phi(M) \quad (\text{the canonical one})$$

$$v\ (\text{«variation»}) : \phi(M) \to \psi(M)$$

induced by a commutative square

$$\begin{array}{ccc} M^\cdot_s & \to & 0 \\ \downarrow{\scriptstyle\varphi} & & \downarrow \\ M^\cdot_\eta & \xrightarrow{N} & M^\cdot_\eta \end{array}$$

So we get a functor

$$M(S)^v \to \left\{\text{triples } \psi \underset{v}{\overset{u}{\rightleftarrows}} \phi, \quad \psi, \phi \in M\right\}$$

5.1.5 THEOREM. *This functor is an equivalence.*

This is the usual description of perserve sheaves over disk, see for example [B2], 3.2.

Proof. An inverse functor takes $(\psi \underset{v}{\overset{u}{\rightleftarrows}} \phi)$ to

$$\begin{array}{ccccccc} 0 & \phi; & (\psi, vu); & \varphi: & \phi & \xrightarrow{v} & \psi \in \tilde{D}(S) \\ & \uparrow{\scriptstyle u} & & , & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle vu} \\ -1 & \psi & & & \psi & = & \psi \end{array}$$

■

5.2 MIXED CATEGORY $\mathcal{O}_\theta$ FOR $g = s\ell_2$.

5.2.1 In notations of BGG2 consider the category $\mathcal{O}_\theta$ for $g = s\ell_2, \theta = \theta_\chi, \chi$ being an integral regular character. It is equivalent to the category whose objects are pairs $A_0, A_1 \in M(\mathbb{C})$ and linear maps $A_0 \underset{v}{\overset{u}{\rightleftarrows}} A_1$ such that $uv = 0$; a morphism $f : A \to A'$ is a pair $f_i \in \mathrm{Hom}(A_i, A'_i), i = 0, 1$, such that $f_i u = u' f_0$; $f_0 v = v' f_1$.

5.2.2 Geometrically, $\mathcal{O}_\theta$ is equivalent to the category of perverse sheaves over $P^1_{\mathbb{C}}$ with singularity only at $0 \in P^1_{\mathbb{C}}$ (cf. 5.1).

5.2.3 Now consider a «mixed» version of this category. Namely, let M be the category whose objects are pairs $M = (M^{\mathrm{ev}}, M^{\mathrm{odd}}), M^{\mathrm{ev}} = \underset{i \in \mathbb{Z}}{\oplus} M_{2i}, M^{\mathrm{odd}} = \underset{i \in \mathbb{Z}}{\oplus} M_{2i+1}$ of graded vector spaces together with degree -1 operators $M^{\mathrm{ev}} \underset{v}{\overset{u}{\rightleftarrows}} M^{\mathrm{odd}}$ such that $uv = 0$; in other words

$$M : \ldots \xrightarrow{v} M_{2i} \xrightarrow{u} M_{2i-1} \xrightarrow{v} M_{2i-2} \xrightarrow{u} \ldots, uv = 0.$$

We put $w_{\leq a} M = \underset{i \leq a}{\oplus} M_i$. So, M is a mixed abelian category, a «hybride» of S- and Λ-modules (3.6.5).

On M one has two symmetries:

$\cdot(1) : M \to M$ (Tate twist):$M(1)_1 = M_{i+2}$;

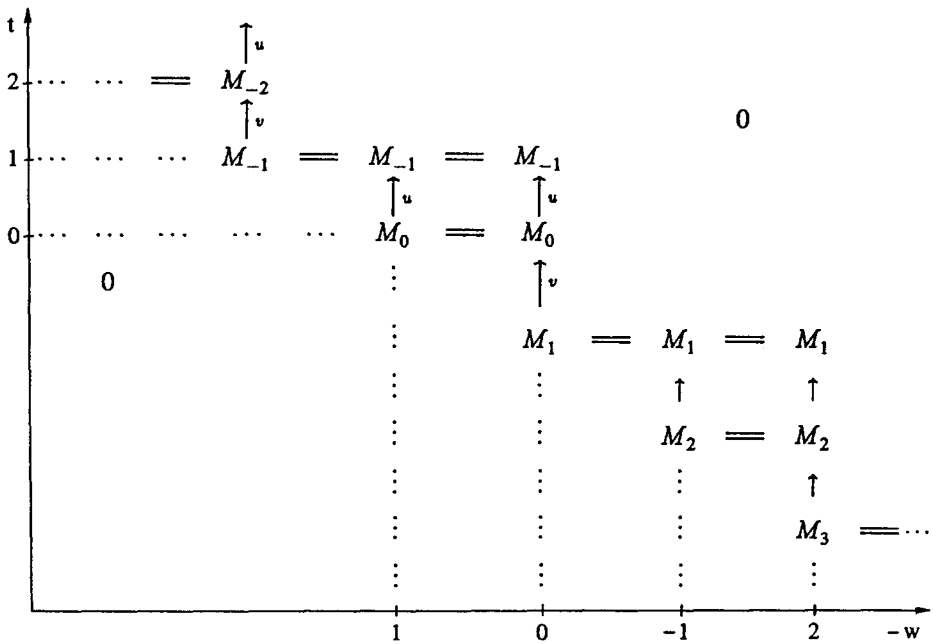
$* : M \tilde{\to} M^{opp}$ («Verdier duality»): $(*M)_i = (M_{-i})^*$,

$$*u = v^*, *v = u^*$$

5.2.4 There exists also a one more hydden symmetry. Namely, it turns out that M is equivalent to its own Koszul dual! More explicitly, put $D = D^b(M)$. One has canonical ft-equivalences

$$^vD \overset{L}{\leftarrow} D \overset{R}{\to} D^v$$

R is induced by the functor $R : M \to C^b(M)$ which takes $M : \ldots \to M_0 \overset{u}{\to} M_{-1} \overset{v}{\to} \ldots$ to



Here the vertical arrow is differential in complex, and horizontal ones are u's and v's of $R(M)$. The functor L is constructed similarly.

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